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Second Edition
CONTINUOUS (ANALOG) AND DISCRETE (DIGITAL)

JOSEPH J. DiSTEFANO, III, Ph.D.
Departments of Computer Science and Medicine
University of California, Los Angeles

ALLEN R. STUBBERUD, Ph.D.
Department of Electrical and Computer Engineering
University of California, Irvine

IVAN J. WILLIAMS, Ph.D.
Space and Technology Group, TRW, Inc.

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Chapter 15

Bode Analysis

15.1 INTRODUCTION

The analysis of feedback control systems using the Bode method is equivalent to Nyquist analysis in that both techniques employ graphical representations of the open-loop frequency response function $GH(\omega)$, where $GH(\omega)$ refers to either a discrete-time or a continuous-time system. However, Bode plots consist of two graphs: the magnitude of $GH(\omega)$, and the phase angle of $GH(\omega)$, both plotted as a function of frequency $\omega$. Logarithmic scales are usually used for the frequency axes and for $|GH(\omega)|$.

Bode plots clearly illustrate the relative stability of a system. In fact, gain and phase margins are often defined in terms of Bode plots (see Example 10.1). These measures of relative stability can be determined for a particular system with a minimum of computational effort using Bode plots, especially for those cases where experimental frequency response data are available.

15.2 LOGARITHMIC SCALES AND BODE PLOTS

Logarithmic scales are used for Bode plots because they considerably simplify their construction, manipulation, and interpretation.

A logarithmic scale is used for the $\omega$-axes (abscissas) because the magnitude and phase angle may be graphed over a greater range of frequencies than with linear frequency axes, all frequencies being equally emphasized, and such graphs for continuous-time systems often result in straight lines (Section 15.4).

The magnitude $|P(\omega)|$ of any frequency response function $P(\omega)$ for any value of $\omega$ is plotted on a logarithmic scale in decibel (db) units, where

$$\text{db} = 20\log_{10}|P(\omega)|$$

[Also see Equation (10.4).]

EXAMPLE 15.1. If $|P(2)| = |GH(2)| = 10$, the magnitude is $20\log_{10}10 = 20$ db.

Since the decibel is a logarithmic unit, the db magnitude of a frequency response function composed of a product of terms is equal to the sum of the db magnitudes of the individual terms. Thus, when the logarithmic scale is employed, the magnitude plot of a frequency response function expressible as a product of more than one term can be obtained by adding the individual db magnitude plots for each product term.

The db magnitude versus log $\omega$ plot is called the Bode magnitude plot, and the phase angle versus log $\omega$ plot is the Bode phase angle plot. The Bode magnitude plot is sometimes called the log-modulus plot in the literature.

EXAMPLE 15.2. The Bode magnitude plot for the continuous-time frequency response function

$$P(j\omega) = \frac{100[1 + j(\omega/10)]}{1 + j\omega}$$

may be obtained by adding the Bode magnitude plots for: 100, $1 + j(\omega/10)$, and $1/(1 + j\omega)$. 

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15.3 THE BODE FORM AND THE BODE GAIN FOR CONTINUOUS-TIME SYSTEMS

It is convenient to use the so-called Bode form of a continuous-time frequency response function when using Bode plots for analysis and design because of the asymptotic approximations in Section 15.4.

The Bode form for the function
\[ K(j\omega + z_1)(j\omega + z_2) \cdots (j\omega + z_m)(j\omega + p_1)(j\omega + p_2) \cdots (j\omega + p_n) \]
where \( l \) is a nonnegative integer, is obtained by factoring out all \( z_i \) and \( p_i \) and rearranging it in the form
\[
\left[ K \prod_{i=1}^{m} z_i / \prod_{i=1}^{n} p_i \right] (1 + j\omega/z_1)(1 + j\omega/z_2) \cdots (1 + j\omega/z_m)(1 + j\omega/p_1)(1 + j\omega/p_2) \cdots (1 + j\omega/p_n) \]

The Bode gain \( K_B \) is defined as the coefficient of the numerator in Equation (15.2):
\[
K_B = \frac{K \prod_{i=1}^{m} z_i}{\prod_{i=1}^{n} p_i} \]  \hspace{1cm} (15.3)

15.4 BODE PLOTS OF SIMPLE CONTINUOUS-TIME FREQUENCY RESPONSE FUNCTIONS AND THEIR ASYMPTOTIC APPROXIMATIONS

The constant \( K_B \) has a magnitude \(|K_B|\), a phase angle of 0° if \( K_B \) is positive, and −180° if \( K_B \) is negative. Therefore the Bode plots for \( K_B \) are simply horizontal straight lines as shown in Figs. 15-1 and 15-2.

The frequency response function (or sinusoidal transfer function) for a pole of order \( l \) at the origin is
\[
\frac{1}{(j\omega)^l} \]  \hspace{1cm} (15.4)

The bode plots for this function are straight lines, as shown in Figs. 15-3 and 15-4.
For a zero of order \( l \) at the origin,

\[
(j\omega)^l.
\]  

(15.5)

the Bode plots are the reflections about the 0-db and 0° lines of Figs. 15-3 and 15-4, as shown in Figs. 15-5 and 15-6.
Consider the single-pole transfer function $p/(s + p)$, $p > 0$. The Bode plots for its frequency response function

$$
\frac{1}{1 + j\omega/p}
$$

are given in Figs. 15-7 and 15-8. Note that the logarithmic frequency scale is normalized in terms of $p$. 
To determine the asymptotic approximations for these Bode plots, we see that for \( \omega/p \ll 1 \), or \( \omega \ll p \),

\[
20 \log_{10} \left| \frac{1}{1 + j\omega/p} \right| \approx 20 \log_{10} 1 = 0 \text{ db}
\]

and for \( \omega/p \gg 1 \), or \( \omega \gg p \),

\[
20 \log_{10} \left| \frac{1}{1 + j\omega/p} \right| \approx 20 \log_{10} \left| \frac{1}{j\omega/p} \right| = -20 \log_{10} \left( \frac{\omega}{p} \right)
\]
Therefore the Bode magnitude plot asymptotically approaches a horizontal straight line at 0 db as $\omega/p$ approaches zero and $-20\log_{10}(\omega/p)$ as $\omega/p$ approaches infinity (Fig. 15-7). Note that this high-frequency asymptote is a straight line with a slope of $-20$ db/decade, or $-6$ db/ octave when plotted on a logarithmic frequency scale as shown in Fig. 15-7. The two asymptotes intersect at the corner frequency $\omega = p$ rad/sec. To determine the phase angle asymptote, we see that for $\omega/p \ll 1$, or $\omega \ll p$,

$$\arg\left(\frac{1}{1+j\omega/p}\right) = -\tan^{-1}\left(\frac{\omega}{p}\right) \equiv 0^\circ$$

and for $\omega/p \gg 1$, or $\omega \gg p$,

$$\arg\left(\frac{1}{1+j\omega/p}\right) = -\tan^{-1}\left(\frac{\omega}{p}\right) \equiv -90^\circ$$

Thus the Bode phase angle plot asymptotically approaches $0^\circ$ as $\omega/p$ approaches zero, and $-90^\circ$ as $\omega/p$ approaches infinity, as shown in Fig. 15-8. A negative-slope straight-line asymptote can be used to join the $0^\circ$ asymptote and the $-90^\circ$ asymptote by drawing a line from the $0^\circ$ asymptote at $\omega = p/5$ to the $-90^\circ$ asymptote at $\omega = 5p$. Note that it is tangent to the exact curves at $\omega = p$.

The errors introduced by these asymptotic approximations are shown in Table 15-1 for the single-pole transfer function at various frequencies.

<table>
<thead>
<tr>
<th>Table 15-1. Asymptotic Errors for $\frac{1}{1+j\omega/p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td>Magnitude error (db)</td>
</tr>
<tr>
<td>Phase angle error</td>
</tr>
</tbody>
</table>

The Bode plots and their asymptotic approximations for the single-zero frequency response function

$$\frac{1}{1 + \frac{j\omega}{z_1}} \quad (15.7)$$

are shown in Figs. 15-9 and 15-10.
Fig. 15-10

Phase angle

\[ \arg(1 + j\omega/\zeta) \]

Asymptotic curve

Normalized frequency, \( \omega/\zeta \)

Fig. 15-11

\[ 20 \log_{10} \frac{1}{1 + j2\omega/\zeta - (\omega/\zeta)^2} \]

db magnitude

Normalized frequency, \( \omega/\omega_n \)
The Bode plots and their asymptotic approximations for the second-order frequency response function with complex poles,

\[ \frac{1}{1 + j2\zeta \omega_n / \omega - \left( \omega / \omega_n \right)^2} \quad 0 \leq \zeta \leq 1 \]  

are shown in Figs. 15-11 and 15-12. Note that the damping ratio \( \zeta \) is a parameter on these graphs.

The magnitude asymptote shown in Fig. 15-11 has a corner frequency at \( \omega = \omega_n \) and a high-frequency slope twice that of the asymptote for the single-pole case of Fig. 15-7. The phase angle asymptote is similar to that of Fig. 15-8 except that the high-frequency portion is at \(-180^\circ\) instead of \(-90^\circ\) and the point of tangency, or inflection, is at \(-90^\circ\).

The Bode plots for a pair of complex zeros are the reflections about the 0 db and 0° lines of those for the complex poles.

### 15.5 CONSTRUCTION OF BODE PLOTS FOR CONTINUOUS-TIME SYSTEMS

Bode plots of continuous-time frequency response functions can be constructed by summing the magnitude and phase angle contributions of each pole and zero (or pairs of complex poles and zeros). The asymptotic approximations of these plots are often sufficient. If more accurate plots are desired, many software packages are available for rapidly accomplishing this task.

For the general open-loop frequency response function

\[ GH(j\omega) = \frac{K_p(1 + j\omega/z_1)(1 + j\omega/z_2) \cdots (1 + j\omega/z_m)}{(j\omega)^l(1 + j\omega/p_1)(1 + j\omega/p_2) \cdots (1 + j\omega/p_n)} \]  

where \( l \) is a positive integer or zero, the magnitude and phase angle are given by

\[ 20 \log_{10} |GH(j\omega)| = 20 \log_{10} |K_p| + 20 \log_{10} \left| \frac{1}{z_1} \right| + \cdots + 20 \log_{10} \left| \frac{1}{z_m} \right| + 20 \log_{10} \left| \frac{1}{p_1} \right| + \cdots + 20 \log_{10} \left| \frac{1}{p_n} \right| \]  

\[ + 20 \log_{10} \left| \frac{1}{(j\omega)^l} \right| + 20 \log_{10} \left| \frac{1}{1 + j\omega/p_1} \right| + \cdots + 20 \log_{10} \left| \frac{1}{1 + j\omega/p_n} \right| \]  

(15.10)
and

\[
\arg GH(j\omega) = \arg K_B + \arg \left(1 + \frac{j\omega}{z_1}\right) + \cdots + \arg \left(1 + \frac{j\omega}{z_m}\right) \\
+ \arg \left(\frac{1}{1+j\omega/P_1}\right) + \cdots + \arg \left(\frac{1}{1+j\omega/P_n}\right) \tag{15.11}
\]

The Bode plots for each of the terms in Equations (15.10) and (15.11) were given in Figs. 15-1 to 15-12. If \(GH(j\omega)\) has complex poles or zeros, terms having a form similar to Equation (15.8) are simply added to Equations (15.10) and (15.11). The construction procedure is best illustrated by an example.

**EXAMPLE 15.3.**  The asymptotic Bode plots for the frequency response function

\[
GH(j\omega) = \frac{10(1+j\omega)}{(j\omega)^2\left[1+j\omega/4 - (\omega/4)^2\right]}
\]

![Fig. 15-13](image1)

![Fig. 15-14](image2)
are constructed using Equations (15.10) and (15.11):

\[ 20 \log_{10} |GH(j\omega)| = 20 \log_{10} 10 + 20 \log_{10} |1 + j\omega| + 20 \log_{10} \left| \frac{1}{(j\omega)^2 + 1 + j\omega/4 - (\omega/4)^2} \right| \]

\[ \arg GH(j\omega) = \arg(1 + j\omega) + \arg\left(\frac{1}{1 + j\omega/4 - (\omega/4)^2}\right) \]

The graphs for each of the terms in these equations are obtained from Figs. 15-1 to 15-12 and are shown in Figs. 15-13 and 15-14. The asymptotic Bode plots for \( GH(j\omega) \) are obtained by adding these curves, as shown in Figs. 15-15 and 15-16, where computer-generated Bode plots for the frequency response function are also given for comparison with the asymptotic approximations.

---

**15.6 BODE PLOTS OF DISCRETE-TIME FREQUENCY RESPONSE FUNCTIONS**

The factored form for the general open-loop discrete-time frequency response function is

\[ GH(e^{j\omega T}) = \frac{K(e^{j\omega T} + z_1)(e^{j\omega T} + z_2) \cdots (e^{j\omega T} + z_m)}{(e^{j\omega T} + p_1)(e^{j\omega T} + p_2) \cdots (e^{j\omega T} + p_n)} \]  \hspace{1cm} (15.12)
Simple asymptotic approximations, similar to those in Section 15.4, do not exist for the individual terms in Equation (15.12). Thus there is no particular advantage to a Bode form of the type in Equation (15.2) for discrete-time systems. In general, computers provide the most convenient way to generate Bode plots for discrete-time systems and several software packages exist to accomplish this task.

For the general open-loop frequency response function Equation (15.12), the magnitude and phase angle are given by

\[
20 \log_{10} |GH(e^{j\omega T})| = 20 \log_{10} |K| + 20 \log_{10} |e^{j\omega T} + z_1| + \cdots + 20 \log_{10} |e^{j\omega T} + z_m| + 20 \log_{10} \frac{1}{|e^{j\omega T} + p_1|} + \cdots + 20 \log_{10} \frac{1}{|e^{j\omega T} + p_n|} \quad (15.13)
\]

and

\[
\arg GH(e^{j\omega T}) = \arg K + \arg (e^{j\omega T} + z_1) + \cdots + \arg \left(\frac{1}{e^{j\omega T} + p_1}\right) + \cdots + \arg \left(\frac{1}{e^{j\omega T} + p_n}\right) \quad (15.14)
\]

It is important to note that both the magnitude and phase angle of discrete-time frequency response functions are periodic in the real angular frequency variable \(\omega\). This is true since

\[e^{j\omega T} = e^{j(\omega + 2k\pi/T)T} = e^{j\omega T}e^{2k\pi} \]

thus \(e^{j\omega T}\) is periodic in the frequency domain with period \(2\pi/T\). Every term in both the magnitude and phase angle is thus periodic. It is therefore only necessary to generate Bode plots over the angular range \(-\pi \leq \omega T \leq \pi\) radians; and the magnitude and phase angle are typically plotted as a function of the angle \(\omega T\) rather than angular frequency \(\omega\).

Another useful property of a discrete-time frequency response function is that the magnitude is an even function of the frequency \(\omega\) (and \(\omega T\)) and the phase angle is an odd function of \(\omega\) (and \(\omega T\)).

**EXAMPLE 15.4.** The Bode plots for the discrete-time frequency response function

\[GH(e^{j\omega T}) = \frac{\frac{1}{30}(e^{j\omega T} + 1)^2}{(e^{j\omega T} - 1)(e^{j\omega T} + \frac{1}{3})(e^{j\omega T} + \frac{1}{2})}\]

are shown in Figs. 15-17 and 15-18.
15.7 RELATIVE STABILITY

The relative stability indicators "gain margin" and "phase margin" for both discrete-time and continuous-time systems are defined in terms of the system open-loop frequency response in Section 10.4. Consequently these parameters are easily determined from the Bode plots of $G_H(\omega)$ as illustrated in Example 10.1, and in Example 15.4 above. Since 0 db corresponds to a magnitude of 1, the gain margin is the number of decibels that $|G_H(\omega)|$ is below 0 db at the phase crossover frequency $\omega_c$ (arg $G_H(\omega_c) = -180^\circ$). The phase margin is the number of degrees arg $G_H(\omega)$ is above $-180^\circ$ at the gain crossover frequency $\omega_1$ ($|G_H(\omega_1)| = 1$). Computer-generated Bode plots should be used to accurately determine $\omega_c$, $\omega_1$ and the gain and phase margins.

In most cases positive gain and phase margins, as defined above, will ensure stability of the closed-loop system. However, a Nyquist Stability Plot (Chapter 11) may be sketched, or one of the methods of Chapter 5 can be used to verify the absolute stability of the system.

EXAMPLE 15.5. The continuous-time system whose Bode plots are shown in Fig. 15-19 has a gain margin of 8 db and a phase margin of $40^\circ$. 
**EXAMPLE 15.6.** For the system in Example 15.4, the gain margin is 39 db, the angle at the phase crossover frequency \( \omega_c \) is \( \omega_c T = 1.57 \text{ rad} \), the phase margin is 90°, and the angle at the gain crossover frequency \( \omega_1 \) is \( \omega_1 T = 0.02 \text{ rad} \), all as illustrated in Figures 15-17 and 15-18.

### 15.8 CLOSED-LOOP FREQUENCY RESPONSE

Although there is no straightforward method for plotting the closed-loop frequency response \((C/R)(\omega)\) from Bode plots of \(GH(\omega)\), it may be approximated in the following manner, for both continuous and discrete-time control systems. The closed-loop frequency response is given by

\[
\frac{C}{R}(\omega) = \frac{G(\omega)}{1 + GH(\omega)}
\]

If \(|GH(\omega)| \gg 1\),

\[
\frac{C}{R}(\omega) \bigg|_{|GH(\omega)| \gg 1} \approx \frac{G(\omega)}{GH(\omega)} = \frac{1}{H(\omega)}
\]

If \(|GH(\omega)| \ll 1\),

\[
\frac{C}{R}(\omega) \bigg|_{|GH(\omega)| \ll 1} \approx G(\omega)
\]

The open-loop frequency response of most systems is characterized by high gain for low frequencies and decreasing gain for higher frequencies, due to the usual excess of poles over zeros. Thus the closed-loop frequency response for a unity feedback system \((H = 1)\) is approximated by a magnitude of 1 (0 db) and phase angle of 0° for frequencies below the gain crossover frequency \(\omega_1\). For frequencies above \(\omega_1\), the closed-loop frequency response may be approximated by the magnitude and phase angle of \(G(\omega)\). An approximate closed-loop bandwidth for many systems is the gain crossover frequency \(\omega_1\) (See Example 12.7.)

**EXAMPLE 15.7.** The open-loop Bode magnitude plot and approximate closed-loop Bode magnitude plot for the continuous-time unity feedback system represented by \(G(j\omega) = 10/j\omega(1 + j\omega)\) are shown in Fig. 15-20.
15.9 BODE ANALYSIS OF DISCRETE-TIME SYSTEMS USING THE \( w \)-TRANSFORM

The \( w \)-transform discussed in Section 10.7 can be used in the Bode analysis of discrete-time systems. The algorithm for Bode analysis using the \( w \)-transform is:

1. Substitute \( (1 + w)/(1 - w) \) for \( z \) in the open-loop transfer function \( GH(z) \):

   \[
   GH(z)_{|z=\frac{1+w}{1-w}} = GH'(w)
   \]

2. Let \( w = j\omega_w \) and generate Bode plots for \( GH'(j\omega_w) \), using the methods of Sections 15.3 through 15.5.

3. Analyze the relative stability of the system in the \( w \)-plane by determining the gain and phase margins, the gain and phase crossover frequencies, the closed-loop frequency response, the bandwidth, and/or any other frequency-related characteristics of interest.

4. Transform the critical frequencies determined in step 3 to the frequency domain of the \( z \)-plane using the transformation \( \omega T = 2 \tan^{-1} \omega_w \).

**EXAMPLE 15.8.** The open-loop transfer function

\[
GH(z) = \frac{\frac{1}{100}(z + 1)^2}{(z - 1)(z + \frac{1}{2})(z + \frac{1}{3})}
\]

is transformed into the \( w \)-domain by letting

\[
z = \frac{1 + w}{1 - w}
\]

which yields

\[
GH'(w) = \frac{-\frac{1}{100}(w - 1)}{w(w + 2)(w + 3)}
\]

Note, in particular, that the minus sign contributes \(-180^\circ \) of phase angle, and the zero at \(+1\) contributes \(+90^\circ \) at \( \omega_w = 0^\circ \). The Bode plots of \( GH'(j\omega_w) \) are shown in Figs. 15-21 and 15-22.
EXAMPLE 15.9. From the Bode plots of Example 15.8, the gain margin in the $w$-domain is 39 db and the phase crossover frequency is $\omega_c = 1$ rad/sec. Transforming back to the $z$-domain, the phase crossover frequency $\omega_c$ is obtained from

$$\omega_c T = 2 \tan^{-1} \omega_c = 1.57 \text{ rad}$$

Compare these results with those of Example 15.6, which are the same.

EXAMPLE 15.10. From the Bode plots of Example 15.8, the phase margin is 90° and the gain crossover frequency is $\omega_{c1} = 0.01$ rad/sec. Transforming to the $z$-domain, the gain crossover frequency $\omega_{c1}$ is obtained from

$$\omega_{c1} T = 2 \tan^{-1} \omega_{c1} = 0.02 \text{ rad}$$

Compare these results with those of Example 15.6, which are the same.

With the wide availability of software for control systems analysis, use of the $w$-transform for Bode analysis of discrete-time systems is usually unnecessary. However, for design by analysis, as discussed in Chapter 16 where insight gained from continuous-time system design techniques is transferred to discrete-time system design, the $w$-transform can be a very useful tool.

**Solved Problems**

LOGARITHMIC SCALES

15.1. Express the following quantities in decibel (db) units: (a) 2, (b) 4, (c) 8, (d) 20, (e) 25, (f) 140.

From Equation (15.1),

- $db_a = 20 \log_{10} 2 = 20(0.301) = 6.02$
- $db_b = 20 \log_{10} 4 = 20(0.602) = 12.04$
- $db_c = 20 \log_{10} 8 = 20(0.903) = 18.06$
- $db_d = 20 \log_{10} 20 = 20(1.301) = 26.02$
- $db_e = 20 \log_{10} 25 = 20(1.398) = 27.96$
- $db_f = 20 \log_{10} 140 = 20(2.146) = 42.92$
Note that since $4 = 2 \times 2$, then for part (b) we have

$$20 \log_{10} 4 = 20 \log_{10} 2 + 20 \log_{10} 2 = 12.04$$

and since $8 = 2 \times 4$, then for part (c) we have

$$20 \log_{10} 8 = 20 \log_{10} 2 + 20 \log_{10} 4 = 6.02 + 12.04 = 18.06$$

THE BODE FORM AND THE BODE GAIN FOR CONTINUOUS-TIME SYSTEMS

15.2. Determine the Bode form and the Bode gain for the transfer function

$$GH = \frac{K(s + 2)}{s^2(s + 4)(s + 6)}$$

Factoring 2 from the numerator, 4 and 6 from the denominator and putting $s = j\omega$ results in the Bode form

$$GH(j\omega) = \frac{(K/12)(1 + j\omega/2)}{(j\omega)^2(1 + j\omega/4)(1 + j\omega/6)}$$

The Bode gain is $K_n = K/12$.

15.3. When is the Bode gain equal to the d.c. gain (zero frequency magnitude) of a transfer function?

The Bode gain is equal to the d.c. gain of any transfer function with no poles or zeros at the origin [$l = 0$ in Equation (15.2)].

BODE PLOTS OF SIMPLE FREQUENCY RESPONSE FUNCTIONS

15.4. Prove that the Bode Magnitude plot for $(j\omega)^l$ is a straight line.

The Bode magnitude plot for $(j\omega)^l$ is a plot of $20 \log_{10} |(j\omega)^l|$ versus $\log_{10} \omega$. Thus

$$\text{slope} = \frac{d(20 \log_{10} |(j\omega)^l|)}{d(\log_{10} \omega)} = 20l \frac{d(\log_{10} |\omega|^l)}{d(\log_{10} \omega)} = 20l$$

Since the slope is constant for any $l$, the Bode magnitude plot is a straight line.

15.5. Determine: (1) the conditions under which the Bode magnitude plot for a pair of complex poles has a peak at a nonzero, finite value of $\omega$; and (2) the frequency at which the peak occurs.

The Bode magnitude is given by

$$20 \log_{10} \left| \frac{1}{1 + j2\xi \omega / \omega_n - (\omega / \omega_n)^2} \right|$$

Since the logarithm is a monotonically increasing function, the magnitude in decibels has a peak (maximum) if and only if the magnitude itself is maximum. The magnitude squared, which is maximum when the magnitude is maximum, is

$$\frac{1}{\left[1 - (\omega / \omega_n)^2\right]^2 + 4(\xi \omega / \omega_n)^2}$$
Taking the derivative of this function and setting it equal to zero yields

\[
\frac{(4\omega/\omega_n^2)[1 - (\omega/\omega_n)^2] - 8\xi^2\omega/\omega_n^2}{\left\{1 - (\omega/\omega_n)^2\right\}^2 + 4(\xi\omega/\omega_n)^2} = 0
\]

or

\[
1 - \left(\frac{\omega}{\omega_n}\right)^2 - 2\xi^2 = 0
\]

and the frequency at the peak is \(\omega = \omega_n\sqrt{1 - 2\xi^2}\). Since \(\omega\) must be real, by definition, the magnitude has a peak at a nonzero value \(\omega\) only if \(1 - 2\xi^2 > 0\) or \(\xi < 1/\sqrt{2} = 0.707\). For \(\xi \geq 0.707\), the Bode magnitude is monotonically decreasing.

**CONSTRUCTION OF BODE PLOTS FOR CONTINUOUS-TIME SYSTEMS**

15.6. Construct the asymptotic Bode plots for the frequency response function

\[
GH(j\omega) = \frac{1 + j\omega/2 - (\omega/2)^2}{j\omega(1 + j\omega/0.5)(1 + j\omega/4)}
\]

The asymptotic Bode plots are determined by summing the graphs of the asymptotic representations for each of the terms of \(GH(j\omega)\), as in equations (15.10) and (15.11). The asymptotes for each of these terms are shown in Figs. 15-23 and 15-24 and the asymptotic Bode plots for \(GH(j\omega)\) in Figs. 15-25 and 15-26. The exact Bode plots generated by computer are shown for comparison.

![Bode Plot Diagram](image-url)  

*Fig. 15-23*
15.7. Construct Bode plots for the frequency response function

\[ GH(j\omega) = \frac{2}{j\omega(1+j\omega/2)(1+j\omega/5)} \]

The asymptotic Bode plots are constructed by summing the asymptotic plots for each term of \( GH(j\omega) \), as in Equation (15.10) and (15.11), and are shown in Figs. 15-27 and 15-28. More accurate curves determined numerically by computer are also plotted for comparison.
15.8. Construct the Bode plots for the open-loop transfer function \( GH = 2(s + 2)/(s^2 - 1) \).

With \( s = j\omega \), the Bode form for this transfer function is

\[
GH(j\omega) = \frac{-4(1 + j\omega/2)}{(1 + j\omega)(1 - j\omega)}
\]

This function has a right-half plane pole [due to the term \( 1/(1 - j\omega) \)] which is not one of the standard functions introduced in Section 15.4. However, this function has the same magnitude as \( 1/(1 + j\omega) \) and the same phase angle as \( 1 + j\omega \). Thus for a function of the form \( 1/(1 - j\omega/p) \), the magnitude can be determined from Fig. 15-7 and the phase angle from Fig. 15-10. For this problem the phase angle contributions from the terms \( 1/(1 + j\omega) \) and \( 1/(1 - j\omega) \) cancel each other. The asymptotes for the Bode magnitude plot are shown in Fig. 15-29 along with a more accurate Bode magnitude plot. The Bode phase angle is determined solely from \( \arg K_B = \arg(-4) - 180^\circ \) and the zero at \( \omega = 2 \), as shown in Fig. 15-30.
RELATIVE STABILITY

15.9. For the system with the open-loop transfer function of Problem 15.6, find \( \omega_1, \omega_m, \) the gain margin, and the phase margin.

Using the exact magnitude curve shown in Fig. 15-25, the gain crossover frequency is \( \omega_1 = 0.62. \) The phase crossover frequency \( \omega_m \) is indeterminate because \( \text{arg} \, GH(j\omega) \) never crosses \(-180^\circ. \) (See Fig. 15-26.) \( \text{Arg} \, GH(j\omega_1) = \text{arg} \, GH(0.62) \) is \(-129^\circ. \) Hence the phase margin is \(-129^\circ + 180^\circ = 51^\circ. \) Since \( \omega_m \) is indeterminate, the gain margin is also indeterminate.

15.10. Determine the gain and phase margins for the systems with the open-loop frequency response function of Problem 15.7.

From Fig. 15-27, \( \omega_1 = 1.5; \) and from Fig. 15-28, \( \text{arg} \, GH(j\omega_1) = -144^\circ. \) Therefore the phase margin is \( 180^\circ - 144^\circ = 36^\circ. \) From Fig. 15-28, \( \omega_m = 3.2; \) and the gain margin is read from Fig. 15-27 as \(-20\log_{10}|GH(j\omega)| = 11 \text{ db.}\)

15.11. Determine the gain and phase margins for the system with the open-loop transfer function of Problem 15.8.

From Fig. 15-29, \( \omega_1 = 2.3 \text{ rad/sec.} \) From Fig. 15-30, \( \text{arg} \, GH(j\omega_1) = -127^\circ. \) Hence the phase margin is \( 180^\circ - 127^\circ = 53^\circ. \) As shown in Fig. 15-30, \( \text{arg} \, GH(j\omega) \) approaches \(-180^\circ \) as \( \omega \) decreases. Since \( \text{arg} \, GH(j\omega) = -180^\circ \) only at \( \omega = 0, \) then \( \omega_m = 0. \) Therefore the gain margin is \(-20\log_{10}|GH(j\omega)| = -12 \text{ db.} \) Using the normal procedure. Although a negative gain margin indicates instability for most systems, this system is stable, as verified by the Nyquist Stability Plot shown in Fig. 15-31. Remember that the system has an open-loop right-half plane pole; but the zero of \( GH \) at \(-2 \) acts to stabilize the system for \( K = 2. \)

![Fig. 15-31](image)

CLOSED-LOOP FREQUENCY RESPONSE

15.12. For the system of Example 15.7 with \( H = 1, \) determine the closed-loop frequency response function and compare the actual closed-loop Bode magnitude plot with the approximate one of Example 15.7.

For this system, \( GH = 10/s(s + 1). \) Then

\[
\frac{C}{R} = \frac{10}{s^2 + s + 10}
\]

and

\[
\frac{C}{R} \left( j\omega \right) = \frac{1}{1 + j\omega/10 - \omega^2/10}
\]

Therefore the closed-loop Bode magnitude plot corresponds to Fig. 15-11, with \( \xi = 0.18 \) and \( \omega_n = 3.16. \) From this plot the actual 3-db bandwidth is \( \omega/\omega_n = 1.5 \) in normalized form; hence, since \( \omega_n = 3.16, \) \( BW = 1.5(3.16) = 4.74 \text{ rad/sec.} \) The approximate 3-db bandwidth determined from Fig. 15-20 of Example
15.7 is 3.7 rad/sec. Note that $\omega_n = 3.16$ rad/sec for the closed-loop system corresponds very well with $\omega_l = 3.1$ rad/sec from Fig. 15-20. Thus the gain crossover frequency of the open-loop system corresponds very well with $\omega_n$ of the closed-loop system, although the approximate 3-db bandwidth determined above is not very accurate. The reason for this is that the approximate Bode magnitude plot of Fig. 15-20 does not show the peaking that occurs in the exact curve.

15.13. For the discrete-time system with open-loop frequency response function

$$GH(z) = \frac{3(z + 1)(z + \frac{1}{2})}{8z(z - 1)(z + \frac{1}{2})}, \quad H = 1$$

find the gain margin, phase margin, phase crossover angle, and gain crossover angle.

The Bode plots for this system are given in Figs. 15-32 and 15-33. The phase crossover angle $\omega_c T$ is determined from Fig. 15-33 as 1.74 rad. The corresponding gain margin is found on Fig. 15-32 as 11 db. The gain crossover angle $\omega_c T$ is determined from Fig. 15-32 as 0.63 rad. The corresponding phase margin is found on Fig. 15-33 as 57°.
Supplementary Problems

15.14. Construct the Bode plots for the open-loop frequency response function

\[ GH(j\omega) = \frac{4(1 + j\omega/2)}{(j\omega)^2(1 + j\omega/8)(1 + j\omega/10)} \]

15.15. Construct the Bode plots and determine the gain and phase margins for the system with the open-loop frequency response function

\[ GH(j\omega) = \frac{4}{(1 + j\omega)(1 + j\omega/3)^2} \]

