Chapter 10

Analysis and Design of Feedback Control Systems: Objectives and Methods

10.1 INTRODUCTION

The basic concepts, mathematical tools, and properties of feedback control systems have been presented in the first nine chapters. Attention is now focused on our major goal: analysis and design of feedback control systems.

The methods presented in the next eight chapters are linear techniques, applicable to linear models. However, under appropriate circumstances, one or more can also be used for some nonlinear control system problems, thereby generating approximate designs when the particular method is sufficiently robust. Techniques for solving control system problems represented by nonlinear models are introduced in Chapter 19.

This chapter is mainly devoted to making explicit the objectives and to describing briefly the methodology of analysis and design. It also includes one digital system design approach, in Section 10.8, that can be considered independently of the several approaches developed in subsequent chapters.

10.2 OBJECTIVES OF ANALYSIS

The three predominant objectives of feedback control systems analysis are the determination of the following system characteristics:

1. The degree or extent of system stability
2. The steady state performance
3. The transient performance

Knowing whether a system is absolutely stable or not is insufficient information for most purposes. If a system is stable, we usually want to know how close it is to being unstable. We need to determine its relative stability.

In Chapter 3 we learned that the complete solution of the equations describing a system may be split into two parts. The first, the steady state response, is that part of the complete solution which does not approach zero as time approaches infinity. The second, the transient response, is that part of the complete solution which approaches zero (or decays) as time approaches infinity. We shall soon see that there is a strong correlation between relative stability and transient response of feedback control systems.

10.3 METHODS OF ANALYSIS

The general procedure for analyzing a linear control system is the following:

1. Determine the equations or transfer function for each system component.
2. Choose a scheme for representing the system (block diagram or signal flow graph).
3. Formulate the system model by appropriately connecting the components (blocks, or nodes and branches).
4. Determine the system response characteristics.

Several methods are available for determining the response characteristics of linear systems. Direct solution of the system equations may be employed to find the steady state and transient solutions...
10.4 DESIGN OBJECTIVES

The basic goal of control system design is meeting performance specifications. Performance specifications are the constraints put on system response characteristics. They may be stated in any number of ways. Generally they take two forms:

1. Frequency-domain specifications (pertinent quantities expressed as functions of frequency)
2. Time-domain specifications (in terms of time response)

The desired system characteristics may be prescribed in either or both of the above forms. In general, they specify three important properties of dynamic systems:

1. Speed of response
2. Relative stability
3. System accuracy or allowable error

Frequency-domain specifications for both continuous and discrete-time systems are often stated in one or more of the following seven ways. To maintain generality, we define a unified open-loop frequency response function \( GH(\omega) \):

\[
GH(\omega) = \begin{cases} 
GH(j\omega) & \text{for continuous systems} \\
GH(e^{j\omega T}) & \text{for discrete-time systems}
\end{cases}
\]  

(10.1)

1. Gain Margin

Gain margin, a measure of relative stability, is defined as the magnitude of the reciprocal of the open-loop transfer function, evaluated at the frequency \( \omega_c \) at which the phase angle (see chapter 6) is \(-180^\circ\). That is,

\[
\text{gain margin} = \frac{1}{|GH(\omega_c)|} 
\]  

(10.2)

where \( \arg GH(\omega_c) = -180^\circ = -\pi \) radians and \( \omega_c \) is called the phase crossover frequency.

2. Phase Margin \( \phi_{PM} \)

Phase margin \( \phi_{PM} \), a measure of relative stability, is defined as \(180^\circ\) plus the phase angle \( \phi_1 \) of the open-loop transfer function at unity gain. That is,

\[
\phi_{PM} = [180 + \arg GH(\omega_1)] \text{ degrees} 
\]  

(10.3)

where \( |GH(\omega_1)| = 1 \) and \( \omega_1 \) is called the gain crossover frequency.
EXAMPLE 10.1. The gain and phase margins of a typical continuous-time feedback control system are illustrated in Fig. 10-1.

3. Delay Time $T_d$

Delay time $T_d$, interpreted as a frequency-domain specification, is a measure of the speed of response, and is given by

$$T_d(\omega) = -\frac{d\gamma}{d\omega}$$

(10.4)

where $\gamma = \arg(C/R)$. The average value of $T_d(\omega)$ over the frequencies of interest is usually specified.

4. Bandwidth (BW)

Roughly speaking, the bandwidth of a system was defined in Chapter 1 as that range of frequencies over which the system responds satisfactorily.

Satisfactory performance is determined by the application and the characteristics of the particular system. For example, audio amplifiers are often compared on the basis of their bandwidth. An ideal high-fidelity audio amplifier has a flat frequency response from 20 to 20,000 Hz. That is, it has a passband or bandwidth of 19,980 Hz (usually rounded off to 20,000 Hz). Flat frequency response means that the magnitude ratio of output to input is essentially constant over the bandwidth. Hence signals in the audio spectrum are faithfully reproduced by a 20,000-Hz bandwidth amplifier. The magnitude ratio is the absolute value of the system frequency response function.

The frequency response of a high-fidelity audio amplifier is shown in Fig. 10-2. The magnitude ratio is 0.707 of, or approximately 3 db below, its maximum at the cutoff frequencies

$$f_{c1} = 20 \, \text{Hz} \quad f_{c2} = 20,000 \, \text{Hz}$$
"db" is the abbreviation for **decibel**, defined by the following equation:

\[ \text{db} = 20 \log_{10} \text{(magnitude ratio)} \]  

(10.5)

Often the bandwidth of a system is defined as that range of frequencies over which the magnitude ratio does not differ by more than \(-3\text{ db}\) from its value at a specified frequency. But not always. In general, the precise meaning of bandwidth is made clear by the problem description. In any case, bandwidth generally a measure of the speed of response of a system.

The gain crossover frequency \(\omega_1\) defined in Equation (10.3) is often a good approximation for the bandwidth of a closed-loop system.

The notion of signal sampling, and of *uniform sampling time* \(T\), were introduced in Chapters 1 and 2 (especially in Section 2.4), for systems containing both discrete-time and continuous-time signals, and both types of elements, including samplers, hold devices and computers. The value of \(T\) is a design parameter for such systems and its choice is governed by both accuracy and cost considerations. The *sampling theorem* [9, 10] provides an upper bound on \(T\), by requiring the sampling rate to be at least twice that of the highest frequency component \(f_{\text{max}}\) of the sampled signal, that is, \(T \leq \frac{f_{\text{max}}}{2}\). In practice, we might use the cutoff frequency \(f_{c2}\) (as in Fig. 10-2) for \(f_{\text{max}}\), and a practical rule-of-thumb might be to choose \(T\) in the range \(\frac{1}{10f_{c2}} \leq T \leq \frac{1}{6f_{c2}}\). Other design requirements, however, may require even smaller \(T\) values. On the other hand, the largest value of \(T\) consistent with the specifications usually yields the lowest cost for system components.

5. **Cutoff Rate**

The cutoff rate is the frequency rate at which the magnitude ratio decreases beyond the cutoff frequency \(\omega_c\). For example, the cutoff rate may be specified as 6 db/octave. An octave is a factor-of-two change in frequency.

6. **Resonance Peak** \(M_p\)

The resonance peak \(M_p\), a measure of relative stability, is the maximum value of the magnitude of the closed-loop frequency response. That is,

\[ M_p = \max_\omega \left| \frac{C}{R} \right| \]  

(10.6)

7. **Resonant Frequency** \(\omega_p\)

The resonant frequency \(\omega_p\) is the frequency at which \(M_p\) occurs.

**EXAMPLE 10.2.** The bandwidth \(\text{BW}\), cutoff frequency \(\omega_c\), resonance peak \(M_p\), and resonant frequency \(\omega_p\) for an underdamped second-order continuous system are illustrated in Fig. 10-3.

![Fig. 10-3](attachment:fig103.png)
**Time-domain specifications** are customarily defined in terms of unit step, ramp, and parabolic responses. Each response has a steady state and a transient component.

*Steady state performance*, in terms of steady state error, is a measure of system accuracy when a specific input is applied. Figures of merit for steady state performance are, for example, the error constants $K_p$, $K_v$, and $K_i$ defined in Chapter 9.

*Transient performance* is often described in terms of the unit step function response. Typical specifications are:

1. **Overshoot**

   The overshoot is the maximum difference between the transient and steady state solutions for a unit step input. It is a measure of relative stability and is often represented as a percentage of the final value of the output (steady state solution).

   The following four specifications are measures of the speed of response.

2. **Delay Time $T_d$**

   The delay time $T_d$, interpreted as a time-domain specification, is often defined as the time required for the response to a unit step input to reach 50% of its final value.

3. **Rise Time $T_r$**

   The rise time $T_r$ is customarily defined as the time required for the response to a unit step input to rise from 10 to 90 percent of its final value.

4. **Settling Time $T_s$**

   The settling time $T_s$ is most often defined as the time required for the response to a unit step input to reach and remain within a specified percentage (frequently 2 or 5%) of its final value.

5. **Dominant Time Constant**

   The dominant time constant $\tau$, an alternative measure for settling time, is often defined as the time constant associated with the term that dominates the transient response.

   The dominant time constant is defined in terms of the exponentially decaying character of the transient response. For example, for first and second-order underdamped continuous systems, the transient terms have the form $Ae^{-at}$ and $Ae^{-at}\cos(\omega_n t + \phi)$, respectively ($\alpha > 0$). In each case, the decay is governed by $e^{-at}$. The time constant $\tau$ is defined as the time at which the exponent $-\alpha t = -1$, that is, when the exponential reaches 37% of its initial value. Hence $\tau = 1/\alpha$.

   For continuous feedback control systems of order higher than two, the dominant time constant can sometimes be estimated from the time constant of an underdamped second-order system which approximates the higher system. Since

   $\tau \leq \frac{1}{\xi \omega_n}$ \hspace{1cm} (10.7)

   $\xi$ and $\omega_n$ (Chapter 3) are the two most significant figures of merit, defined for second-order but often useful for higher-order systems. Specifications are often given in terms of $\xi$ and $\omega_n$.

   This concept is developed more fully for both continuous and discrete-time systems in Chapter 14, in terms of dominant pole-zero approximations.

**EXAMPLE 10.3.** The plot of the unit step response of an underdamped continuous second-order system in Fig. 10-4 illustrates time-domain specifications.
10.5 SYSTEM COMPENSATION

We assume first that $G$ and $H$ are fixed configurations of components over which the designer has no control. To meet performance specifications for feedback control systems, appropriate compensation components (sometimes called equalizers) are normally introduced into the system. Compensation components may consist of either passive or active elements, several of which were discussed in Chapters 2 and 6. They may be introduced into the forward path (cascade compensation), or the feedback path (feedback compensation), as shown in Fig. 10-5:

Feedback compensation may also occur in minor feedback loops (Fig. 10-6).
Compensators are normally designed so that the overall system (continuous or discrete) has an acceptable transient response, and hence stability characteristics, and a desired or acceptable steady state accuracy (Chapter 9). These objectives are often conflicting, because small steady state errors usually require large open-loop gains, which typically degrade system stability. For this reason, simple compensator elements are often combined in a single design. They typically consist of combinations of components that modify the gain $K$ and/or time constants $\tau$, or otherwise add zeros or poles to $GH$. Passive compensators include passive physical elements such as resistive-capacitive networks, to modify $K$ ($K < 1$), time constants, zeros, or poles; lag, lead, and lag-lead networks are examples (Chapter 6). The most common active compensator is the amplifier ($K > 1$). A very general one is the PID (proportional-integral-derivative) controller discussed in Chapter 2 and 6 (Examples 2.14 and 6.7), commonly used in the design of both analog (continuous) and discrete-time (digital) systems.

10.6 DESIGN METHODS

Design by analysis is the design scheme developed in this book, because it is generally a more practical approach, with the exception that direct design of digital systems, discussed in Section 10.8, is a true synthesis technique. The previously mentioned analysis methods, reiterated below, are applied to design in Chapters 12, 14, 16, and 18.

1. Nyquist Plot (Chapter 12)
2. Root-Locus (Chapter 14)
3. Bode Plot (Chapters 16)
4. Nichols Chart (Chapter 18)

Control system analysis and design procedures based on these methods have been automated in special-purpose computer software packages called Computer-Aided Design (CAD) packages.

Of the four methods listed above, the Nyquist, Bode, and Nichols methods are frequency response techniques, because in each of them the properties of $GH(\omega)$, that is, $GH(j\omega)$ for continuous systems or $GH(e^{j\omega T})$ for discrete-time systems [Equation (10.1)], are explored graphically as a function of angular frequency $\omega$. More importantly, analysis and design using these methods is performed in fundamentally the same manner for continuous and discrete-time systems, as illustrated in subsequent chapters. The only differences (in specific details) stem from the fact that the stability region for continuous systems is the left half of the $s$-plane, and that for discrete-time systems is the unit circle in the $z$-plane. A transformation of variables, however, called the $w$-transform, permits analysis and design of discrete-time systems using specific results developed for continuous systems. We present the major features and the results for the $w$-transform in the next section, for use in analysis and design of control systems in subsequent chapters.

10.7 THE $w$-TRANSFORM FOR DISCRETE-TIME SYSTEMS ANALYSIS AND DESIGN USING CONTINUOUS SYSTEM METHODS

The $w$-transform was defined in Chapter 5 for stability analysis of discrete-time systems. It is a bilinear transformation between the complex $w$-plane and the complex $z$-plane defined by the pair:

$$w = \frac{z - 1}{z + 1}, \quad z = \frac{1 + w}{1 - w} \quad (10.8)$$

where $z = \mu + j\nu$. The complex variable $w$ is defined as

$$w = \text{Re} \, w + j \, \text{Im} \, w \quad (10.9)$$
The following relations among these variables are useful in the analysis and design of discrete-time control systems:

1. \[ \text{Re}\ w = \frac{\mu^2 + \nu^2 - 1}{\mu^2 + \nu^2 + 2\mu + 1} \] (10.10)

2. \[ \text{Im}\ w = \frac{2\nu}{\mu^2 + \nu^2 + 2\mu + 1} \] (10.11)

3. If \(|z| < 1\), then \(\text{Re}\ w < 0\) (10.12)

4. If \(|z| = 1\), then \(\text{Re}\ w = 0\) (10.13)

5. If \(|z| > 1\), then \(\text{Re}\ w > 0\) (10.14)

6. On the unit circle of the \(z\)-plane:

\[ z = e^{j\omega T} = \cos \omega T + j\sin \omega T \] (10.15)

\[ \mu^2 + \nu^2 = \cos^2 \omega T + \sin^2 \omega T = 1 \] (10.16)

\[ w = \frac{j\mu}{\mu + 1} \] (10.17)

Thus the region inside the unit circle in the \(z\)-plane maps into the left half of the \(w\)-plane (LHP); the region outside the unit circle maps into the right half of the \(w\)-plane (RHP); and the unit circle maps onto the imaginary axis of the \(w\)-plane. Also, rational functions of \(z\) map into rational functions of \(w\).

For these reasons, absolute and relative stability properties of discrete systems can be determined using methods developed for continuous systems in the \(s\)-plane. Specifically, for frequency response analysis and design of discrete-time systems in the \(w\)-plane, we generally treat the \(w\)-plane as if it were the \(s\)-plane. However, we must account for distortions in certain mappings, particularly angular frequency, when interpreting the results.

From Equation (10.17), we define an angular frequency \(\omega_w\) on the imaginary axis in the \(w\)-plane by

\[ \omega_w = \frac{\nu}{\mu + 1} \] (10.18)

This new angular frequency \(\omega_w\) in the \(w\)-plane is related to the true angular frequency \(\omega\) in the \(z\)-plane by

\[ \omega_w = \tan -\frac{T}{2} \text{ or } \omega = \frac{2}{T} \tan^{-1} \omega_w \] (10.19)

The following properties of \(\omega_w\) are useful in plotting functions for frequency response analysis in the \(w\)-plane:

1. If \(\omega = 0\), then \(\omega_w = 0\) (10.20)

2. If \(\omega \rightarrow \frac{\pi}{T}\), then \(\omega_w \rightarrow +\infty\) (10.21)

3. If \(\omega \rightarrow -\frac{\pi}{T}\), then \(\omega_w \rightarrow -\infty\) (10.22)

4. The range \(-\frac{\pi}{T} < \omega < \frac{\pi}{T}\) is mapped into the range \(-\infty < \omega_w < +\infty\) (10.23)

**Algorithm for Frequency Response Analysis and Design Using the \(w\)-Transform**

The procedure is summarized as follows:

1. Substitute \((1 + w)/(1 - w)\) for \(z\) in the open-loop transfer function \(GH(z)\):

\[ GH(z)\big|_{z=(1+w)/(1-w)} = GH'(w) \] (10.24)
Compensators are normally designed so that the overall system (continuous or discrete) has an acceptable transient response, and hence stability characteristics, and a desired or acceptable steady state accuracy (Chapter 9). These objectives are often conflicting, because small steady state errors usually require large open-loop gains, which typically degrade system stability. For this reason, simple compensator elements are often combined in a single design. They typically consist of combinations of components that modify the gain $K$ and/or time constants $\tau$, or otherwise add zeros or poles to $GH$. Passive compensators include passive physical elements such as resistive-capacitive networks, to modify $K$ ($K < 1$), time constants, zeros, or poles; lag, lead, and lag-lead networks are examples (Chapter 6). The most common active compensator is the amplifier ($K > 1$). A very general one is the PID (proportional-integral-derivative) controller discussed in Chapter 2 and 6 (Examples 2.14 and 6.7), commonly used in the design of both analog (continuous) and discrete-time (digital) systems.

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where $z = \mu + j\nu$. The complex variable $w$ is defined as

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w = \text{Re} w + j \text{Im} w
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The following relations among these variables are useful in the analysis and design of discrete-time control systems:

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This new angular frequency \(\omega_w\) in the \(w\)-plane is related to the true angular frequency \(\omega\) in the \(z\)-plane by

\[ \omega_w = \tan \left( \frac{\omega T}{2} \right) \quad \text{or} \quad \omega = \frac{2}{T} \tan^{-1} \omega_w \] (10.19)

The following properties of \(\omega_w\) are useful in plotting functions for frequency response analysis in the \(w\)-plane:

1. If \(\omega = 0\), then \(\omega_w = 0\) \hspace{0.5cm} (10.20)

2. If \(\omega \to \frac{\pi}{T}\), then \(\omega_w \to +\infty\) \hspace{0.5cm} (10.21)

3. If \(\omega \to -\frac{\pi}{T}\), then \(\omega_w \to -\infty\) \hspace{0.5cm} (10.22)

4. The range \(-\frac{\pi}{T} < \omega < \frac{\pi}{T}\) is mapped into the range \(-\infty < \omega_w < +\infty\) \hspace{0.5cm} (10.23)

Algorithm for Frequency Response Analysis and Design Using the \(w\)-Transform

The procedure is summarized as follows:

1. Substitute \((1 + w)/(1 - w)\) for \(z\) in the open-loop transfer function \(GH(z)\):

\[ GH(z) \bigg|_{z=(1+w)/(1-w)} = GH'(w) \] (10.24)
2. Generate frequency response curves, that is, Nyquist Plots, Bode Plots, etc., for

\[ GH'(w)|_{w=j\omega} = GH'(j\omega) \] (10.25)

3. Analyze relative stability properties of the system in the \( w \)-plane (as if it were the \( s \)-plane). For example, determine gain and phase margins, crossover frequencies, the closed-loop frequency response, the bandwidth, or any other desired frequency-response-related characteristics.

4. Transform \( w \)-plane critical frequencies (values of \( \omega \)) determined in Step 3 into angular frequencies (values of \( \omega \)) in the true frequency domain (\( z \)-plane), using Equation (10.19).

5. If this is a design problem, design appropriate compensators to modify \( GH'(j\omega) \) to satisfy performance specifications.

This algorithm is developed further and applied in Chapters 15 through 18.

**EXAMPLE 10.4.** The open-loop transfer function

\[ GH(z) = \frac{(z + 1)^2}{100} \frac{z + 1}{z + \frac{1}{2}} \] (10.26)

is transformed into the \( w \)-domain by substituting \( z = \frac{1 + w}{1 - w} \) in the expression for \( GH(z) \), which yields

\[ GH'(w) = \frac{-6(w - 1)/100}{w(w + 2)(w + 3)} \] (10.27)

Relative stability analysis of \( GH'(w) \) is postponed until Chapter 15.

### 10.8 ALGEBRAIC DESIGN OF DIGITAL SYSTEMS, INCLUDING DEADBEAT SYSTEMS

When digital computers or microprocessors are components of a discrete-time system, compensators can be readily implemented in software or firmware, thereby facilitating direct design of the system by algebraic solution for the transfer function of the compensator that satisfies given design objectives. For example, suppose we wish to construct a system having a given closed-loop transfer function \( C/R \), which might be defined by requisite closed-loop characteristics such as bandwidth, steady state gain, response time, etc. Then, given the plant transfer function \( G_2(z) \), the required forward loop compensator \( G_1(z) \) can be determined from the relation for the closed-loop transfer function of the canonical system given in Section 7.5:

\[ \frac{C}{R} = \frac{G_1G_2}{1 + G_1G_2H} \] (10.28)

Then the required compensator is determined by solving for \( G_1(z) \):

\[ G_1 = \frac{C/R}{G_2(1 - HC/R)} \] (10.29)

**EXAMPLE 10.5.** The unity feedback \( (H = 1) \) system in Fig. 10-7, with \( T = 0.1 \text{-sec uniform and synchronous sampling} \), is required to have a steady state gain \( (C/R)(1) = 1 \) and a rise time \( T_r \) of 2 sec or less.

\[ \frac{C}{R} \]

\[ G(z) \]

\[ 1 \]

\[ \frac{1}{z - 0.5} \]

Fig. 10-7
The simplest \( \frac{C}{R} \) that satisfies the requirements is \( (\frac{C}{R}) = 1 \). However, the required compensator would be

\[
G_1 = \frac{C}{G_2 \left(1 - \frac{C}{R}\right)} = \frac{1}{\frac{1}{z-0.5}(1-1)} = \frac{z-0.5}{0}
\]

which has infinite gain, a zero at \( z = 0.5 \), and no poles, which is unrealizable. For realizability (Section 6.6), \( G_1 \) must have at least as many poles as zeros. Consequently, even with cancellation of the poles and zeros of \( G_2 \) by zeros and poles of \( G_1 \), \( \frac{C}{R} \) must contain at least \( n - m \) poles, where \( n \) is the number of poles and \( m \) is the number of zeros of \( G_2 \).

The simplest realizable \( \frac{C}{R} \) has the form:

\[
\frac{C}{R} = \frac{K}{z-a}
\]

As shown in Problem 10.10, the rise time for a first-order discrete-time system, like the one given by \( \frac{C}{R} \) above, is

\[
T_r \leq \frac{T \ln \frac{1}{a}}{\ln a}
\]

Solving for \( a \), we get

\[
a = \left[ \frac{1}{9} \right]^{T_r / T} = \left[ \frac{1}{9} \right]^{20} = 0.8959
\]

Then

\[
\frac{C}{R} = \frac{K}{z-a} = \frac{K}{z-0.8959}
\]

and, for the steady state gain \( (\frac{C}{R})(1) \) to be 1, \( K = 1 - 0.8959 = 0.1041 \). Therefore the required compensator is

\[
G_1 = \frac{C}{G_2 \left(1 - \frac{C}{R}\right)} = \frac{0.1041}{\frac{1}{z-0.5}(1-1)} = \frac{0.1041(z-0.5)}{z-1}
\]

We see that \( G_1 \) has added a pole to \( G_1 G_2 \) at \( z = 1 \), making the system type 1. This is due to the requirement that the steady state gain equal 1.

**Deadbeat systems** are a class of discrete-time systems that can be readily designed using the direct approach described above. By definition, the closed-loop transient response of a deadbeat system has finite length, that is, it becomes zero, and remains zero, after a finite number of sample times. In response to a step input, the output of such a system is constant at each sample time after a finite period. This is termed a deadbeat response.

**EXAMPLE 10.6.** For a unity feedback system with forward transfer function

\[
G_2(z) = \frac{K_1(z+z_1)}{(z+p_1)(z+p_2)}
\]

introduction of a feedforward compensator with

\[
G_1(z) = \frac{(z+p_1)(z+p_2)}{(z-K_1)(z+z_1)}
\]

results in the closed-loop transfer function:

\[
\frac{C}{R} = \frac{G_1 G_2}{1 + G_1 G_2} = \frac{K_1}{z}
\]

The impulse response of this system is \( c(0) = K_1 \) and \( c(k) = 0 \) for \( k > 0 \). The step response is \( c(0) = 0 \) and \( c(k) = K_1 \) for \( k > 0 \).
In general, systems can be designed to exhibit a deadbeat response with a transient response $n - m$ samples long, where $m$ is the number of zeros and $n$ is the number of poles of the plant. However, to avoid intersample ripple (periodic or aperiodic variations) in mixed continuous/discrete-time systems, where $G_2(z)$ has a continuous input and/or output, the zeros of $G_2(z)$ should not be cancelled by the compensator as in Example 10.5. The transient response in these cases is a minimum of $n$ samples in length and the closed loop transfer function has $n$ poles at $z = 0$.

**EXAMPLE 10.7.** For a system with

$$G_2(z) = \frac{K(z + 0.5)}{(z - 0.2)(z - 0.4)}$$

let

$$G_1(z) = \frac{(z - 0.2)(z - 0.4)}{(z + a)(z + b)}$$

Then

$$\frac{C}{R} = \frac{G_1G_2}{1 + G_1G_2} = \frac{K(z + 0.5)}{(z + a)(z + b) + K(z + 0.5)} = \frac{K(z + 0.5)}{z^2 + (a + b + K)z + ab + 0.5K}$$

For a deadbeat response, we choose

$$\frac{C}{R} = \frac{K(z + 0.5)}{z^2}$$

and therefore

$$a + b + K = 0$$

$$ab + 0.5K = 0$$

There are many possible solutions for $a$, $b$, and $K$ and one is $a = 0.3$, $b = -0.75$, and $K = 0.45$.

If it is required that the closed-loop system be type I, it is necessary that $G_1(z)G_2(z)$ contain $I$ poles at $z = 1$. If $G_2(z)$ has the required number of poles, they should be retained, that is, not cancelled by zeros of $G_1(z)$. If $G_2(z)$ does not have all the required poles at $z = 1$, they can be added in $G_1(z)$.

**EXAMPLE 10.8.** For the system with

$$G_2(z) = \frac{K}{z - 1}$$

suppose a type 2 closed-loop system with deadbeat response is desired. This can be achieved with a compensator of the form:

$$G_1(z) = \frac{z + a}{z - 1}$$

which adds a pole at $z = 1$. Then

$$\frac{C}{R} = \frac{G_1G_2}{1 + G_1G_2} = \frac{K(z + a)}{(z - 1)^2 + K(z + a)} = \frac{K(z + a)}{z^2 + (K - 2)z + 1 + Ka}$$

If a deadbeat response is desired, we must have

$$\frac{C}{R} = \frac{K(z + a)}{z^2}$$

and therefore $K - 2 = 0$ and $1 + Ka = 0$, giving $K = 2$ and $a = -0.5$. 
Solved Problems

10.1. The graph of Fig. 10-8 represents the input-output characteristic of a controller-amplifier for a feedback control system whose other components are linear. What is the linear range of $e(t)$ for this system?

![Fig. 10-8](image)

The amplifier-controller operates linearly over the approximate range $-e_3 \leq e \leq e_5$.

10.2. Determine the gain margin for the system in which $\frac{G}{H}(j\omega) = \frac{1}{(j\omega + 1)^3}$.

Writing $\frac{G}{H}(j\omega)$ in polar form, we have

$$\frac{G}{H}(j\omega) = \frac{1}{(\omega^2 + 1)^{3/2}} \sqrt{-3\tan^{-1}\omega} \quad \arg \frac{G}{H}(j\omega) = -3\tan^{-1}\omega$$

Then $-3\tan^{-1}\omega = -\pi$, $\omega_n = \tan(\pi/3) = 1.732$. Hence, by Equation (10.2), gain margin = $1/|\frac{G}{H}(j\omega_n)| = 8$.

10.3. Determine the phase margin for the system of Problem 10.2.

We have

$$|\frac{G}{H}(j\omega)| = \frac{1}{(\omega^2 + 1)^{3/2}} = 1$$

only when $\omega = \omega_1 = 0$. Therefore

$$\phi_{PM} = 180^\circ + (-3\tan^{-1}0) = 180^\circ = \pi \text{ radians}$$

10.4. Determine the average value of $T_d(\omega)$ over the frequency range $0 \leq \omega \leq 10$ for $\frac{C}{R} = j\omega/(j\omega + 1)$. $T_d(\omega)$ is given by Equation (10.4).

$$\gamma = \arg \frac{C}{R}(j\omega) = \frac{\pi}{2} - \tan^{-1}\omega \quad \text{and} \quad T_d(\omega) = -\frac{d\gamma}{d\omega} = \frac{d}{d\omega} \left[ \tan^{-1}\omega \right] = \frac{1}{1 + \omega^2}$$

Therefore

$$\text{Avg } T_d(\omega) = \frac{1}{10} \int_0^{10} \frac{d\omega}{1 + \omega^2} = 0.147 \text{ sec}$$

10.5. Determine the bandwidth for the system with transfer function $\frac{C}{R}(s) = 1/(s + 1)$.

We have

$$\left| \frac{C}{R}(j\omega) \right| = \frac{1}{\sqrt{\omega^2 + 1}}$$

A sketch of $|\frac{C}{R}(j\omega)|$ versus $\omega$ is given in Fig. 10-9.
10.6. How many octaves are between (a) 200 Hz and 800 Hz, (b) 200 Hz and 100 Hz, (c) 10,048 rad/sec (rps) and 100 Hz?
(a) Two octaves.
(b) One octave.
(c) \( f = \omega /2\pi = 10,048 /2\pi = 1600 \) Hz. Hence there are four octaves between 10,048 rps and 100 Hz.

10.7. Determine the resonance peak \( M_p \) and the resonant frequency \( \omega_p \) for the system whose transfer function is \( (C/R)(s) = 5/(s^2 + 2s + 5) \).

\[
\left| \frac{C}{R}(j\omega) \right| = \frac{5}{\sqrt{\omega^2 + 2j\omega + 5}} = \frac{5}{\sqrt{\omega^2 - 6\omega^2 + 25}}
\]

Setting the derivative of \( |(C/R)(j\omega)| \) equal to zero, we get \( \omega_p = \pm \sqrt{3} \). Therefore

\[
M_p = \max_{\omega} \left| \frac{C}{R}(j\omega) \right| = \left| \frac{C}{R}(j\sqrt{3}) \right| = \frac{5}{4}
\]

10.8. The output in response to a unit step function input for a particular continuous control system is \( c(t) = 1 - e^{-t} \). What is the delay time \( T_d \)?

The output is given as a function of time. Therefore, the time-domain definition of \( T_d \) presented in Section 10.4 is applicable. The final value of the output is \( \lim_{t \to \infty} c(t) = 1 \). Hence \( T_d \) (at 50% of the final value) is the solution of \( 0.5 = 1 - e^{-T_d} \), and is equal to \( \log_e(2) \), or 0.693.

10.9. Find the rise time \( T_r \) for \( c(t) = 1 - e^{-t} \).

At 10% of the final value, \( 0.1 = 1 - e^{-t_1} \); hence \( t_1 = 0.104 \) sec. At 90% of the final value, \( 0.9 = 1 - e^{-t_2} \); thus \( t_2 = 2.302 \) sec. Then \( T_r = 2.302 - 0.104 = 2.198 \) sec.

10.10. Determine the rise time of the first-order discrete system

\( P(z) = (1-a)/(z-a) \) with \( |a| < 1 \).

For a step input, the output transform is

\( Y(z) = P(z)U(z) = \frac{(1-a)z}{(z-1)(z-a)} \)

and the time response is \( y(k) = 1 - a^k \) for \( k = 0,1, \ldots \). Since \( y(\infty) = 1 \), the rise time \( T_r \) is the time required for this unit step response to go from 0.1 to 0.9. Since the sampled response may not have the exact values 0.1 and 0.9, we must find the sampled values that bound these values. Thus, for the lower value, \( y(k) \leq 0.1 \), or \( 1 - a^k \leq 0.1 \) and therefore \( a^k \geq 0.9 \). Similarly for \( y(k + T_r/T) = 1 - a^{k+T_r/T} \geq 0.9, a^k+T_r/T \leq 0.1 \).
Dividing the two expressions, we get

$$\frac{a^{k+T/T}}{a^k} \leq \frac{1}{9}$$

or

$$a^{T/T} \leq \frac{1}{9}$$

Then, by taking logarithms of both sides, we get

$$T \leq T \ln \frac{1}{\ln a}$$

10.11. Verify the six properties of the \( w \)-transform in Section 10.7, Equations (10.10) through (10.17).

From \( w = (z-1)/(z+1) \) and \( z = \mu + j \nu \),

$$w = \frac{\mu + j \nu - 1}{\mu + j \nu + 1} = \frac{(\mu - 1 + j \nu)(\mu + 1 - j \nu)}{(\mu + 1 + j \nu)(\mu + 1 - j \nu)} = \left(\frac{\mu^2 + \nu^2 - 1}{\mu^2 + \nu^2 + 2 \mu + 1}\right) + j\left(\frac{2 \nu}{\mu^2 + \nu^2 + 2 \mu + 1}\right)$$

Thus

1. \( \text{Re } w = \frac{\mu^2 + \nu^2 - 1}{\mu^2 + \nu^2 + 2 \mu + 1} = \sigma_w \)
2. \( \text{Im } w = \frac{2 \nu}{\mu^2 + \nu^2 + 2 \mu + 1} = \omega_w \)
3. \( |z| < 1 \) means \( \mu^2 + \nu^2 < 1 \), which implies \( \sigma_w < 0 \)
4. \( |z| = 1 \) means \( \mu^2 + \nu^2 = 1 \), which implies \( \sigma_w = 0 \)
5. \( |z| > 1 \) means \( \mu^2 + \nu^2 > 1 \), which implies \( \sigma_w > 0 \)

The sixth property follows from elementary trigonometric identities.

10.12. Show that the transformed angular frequency \( \omega_w \) is related to the real frequency \( \omega \) by Equation (10.19).

From Problem 10.11, \( |z| = 1 \) also implies that \( w = j[\nu/(\mu + 1)] = j \omega_w \) [Equation (10.17)]. But \( |z| = 1 \) implies that \( z = e^{j \omega T} \) which implies \( \omega = \mu + j \nu \) [Equation (10.15)]. Therefore

$$\omega_w = \frac{\sin \omega T}{\cos \omega T + 1}$$

Finally, substituting the following half-angle identities of trigonometry into the last expression:

$$2 \sin \left(\frac{\omega T}{2}\right) \cos \left(\frac{\omega T}{2}\right) = \sin \omega T$$

$$\cos^2 \left(\frac{\omega T}{2}\right) - \sin^2 \left(\frac{\omega T}{2}\right) = \cos \omega T$$

we have

$$\omega_w = \frac{\sin \left(\frac{\omega T}{2}\right) \cos \left(\frac{\omega T}{2}\right)}{2 \cos^2 \left(\frac{\omega T}{2}\right)} = \tan \left(\frac{\omega T}{2}\right)$$
For the uniformly and synchronously sampled system given in Fig. 10-10, determine $G_1(z)$ so that the system is type 1 with a deadbeat response.

![Diagram](image)

The forward loop z-transform, assuming fictitious sampling of the output $c(t)$ (see Section 6.8), is determined from Equation (6.9):

$$G_2(z) = \left. \frac{z^{-1} \left( G(s) \right)_{r=kT}}{z - 1} \right|_{s = \frac{1}{z}} = \frac{K_1(z + z_i)}{(z - 1)(z - e^{-T})}$$

where

$$K_1 = K(T + e^{-T} - 1) \quad \text{and} \quad z_i = \frac{1 - e^{-T} - Te^{-T}}{T + e^{-T} - 1}$$

Let $G_1(z)$ have the form $G_1(z) = (z - e^{-T})/(z + b)$. Then, if we also assume a fictitious sampler at the input $r(t)$, we can determine the closed-loop z-domain transfer function:

$$\frac{C}{R} = \frac{G_1 G_2}{1 + G_1 G_2} = \frac{K_1(z + z_i)}{(z - 1)(z + b) + K_1(z + z_i)} = \frac{K_1(z + z_i)}{z^2 + (b - 1 + K_1)z - b + K_1 z_i}$$

For a deadbeat response, $b - 1 + K_1 = 0$ $(b = 1 - K_1)$ and $-b + K_1 z_i = 0$ $(-1 + K_1 + K_1 z_i = 0)$. Then

$$K_1 = \frac{1}{1 + z_i}$$

and

$$b - 1 - K_1 = \frac{z_i}{1 + z_i}$$

Since $K_1 = K(T + e^{-T} - 1)$,

$$K = \frac{K_1}{T + e^{-T} - 1} = \frac{1}{(1 + z_i)(T + e^{-T} - 1)} = \frac{1}{T(1 - e^{-T})}$$

For this system, with continuous input and output signals, $(C/R)(z)$ determined above gives the closed-loop input-output relationship at the sampling times only.

**Supplementary Problems**

10.14. Determine the phase margin for $GH = 2(s + 1)/s^2$.

10.15. Find the bandwidth for $GH = 60/s(s + 2)(s + 6)$ for the closed-loop system.

10.16. Calculate the gain and phase margin for $GH = 432/s(s^2 + 13s + 115)$.

10.17. Calculate the phase margin and bandwidth for $GH = 640/(s^4 + 4s^3 + 16)$ for the closed-loop system.
Answers to Supplementary Problems

10.14. $\phi_{PM} = 65.5^\circ$

10.15. $BW = 3$ rad/sec

10.16. Gain margin = 3.4, phase margin = 65$^\circ$

10.17. $\phi_{PM} = 17^\circ$, $BW = 5.5$ rad/sec