INTRODUCTION TO SOLID STATE PHYSICS
Advanced Topic K

JOSEPHSON SUPERCONDUCTOR TUNNELING EFFECTS

Quantum tunneling of single electrons from a superconductor across an insulator and into a superconductor or into a normal metal was discussed in Chapter 12. The results shown there are typical of tunneling experiments unless exceptional precautions are taken in the construction of the junction, as described below.

Under suitable conditions we observe remarkable effects associated with the tunneling of superconducting electron pairs from a superconductor through a layer of an insulator into another superconductor (Fig. 1). The effects of pair tunneling are quite unlike single particle tunneling and include:

**Dc Josephson effect.** A dc current flows across the junction in the absence of any electric or magnetic field.

**Ac Josephson effect.** A dc voltage applied across the junction causes rf current oscillations across the junction. This effect has been utilized in a precision determination of the value of $\hbar/e$. Further, an rf voltage applied with the dc voltage can then cause a dc current across the junction.

![Figure 1 (a)](a) A tunneling junction composed of two superconductors separated by a thin layer of an insulator, which may be an oxide layer of the order of 10 Å in thickness formed on one of the superconductors.

![Figure 1 (b)](b) An actual tunneling junction may be fabricated by evaporation of a rectangular strip of lead on the surface of a glass microscope slide; the strip then is allowed to oxidize, and a second strip of lead is deposited at right angles to the first strip. The resistance of the junction may be of the order of 1 ohm; the contact area of the order of $10^{-4}$ cm$^2$; the maximum Josephson current of the order of 1 mA. The earth's magnetic field has a deleterious dephasing effect across the area of the junction, so that the field may have to be screened out.

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All tunneling is a quantum effect: by tunneling we mean the penetration of a particle through a potential barrier—that is, through a region which is forbidden to it on classical mechanics.
Macroscopic long range quantum interference. A dc magnetic field applied through a superconducting circuit containing two junctions causes the maximum supercurrent to show interference effects as a function of magnetic field intensity. This effect can be utilized in sensitive magnetometers.

Our discussion of the Josephson junction phenomena follows the discussion of Advanced Topic J.

**DC JOSEPHSON EFFECT**

Let $\psi_1$ be the probability amplitude of electron pairs on one side of a junction, and let $\psi_2$ be the amplitude on the other side. For simplicity, let both superconductors be identical. For the present we suppose that they are both at zero potential.

The time-dependent Schrödinger equation $i\hbar \partial \psi / \partial t = \hat{H} \psi$ applied to the two amplitudes gives

$$i\hbar \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2 ; \quad i\hbar \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1 ,$$

(1)

a brilliant oversimplification of the problem. Here $\hbar T$ represents the effect of the electron-pair coupling or transfer interaction across the insulator; $T$ has the dimensions of a rate or frequency. It is a measure of the leakage of $\psi_1$ into the region 2, and of $\psi_2$ into the region 1. If the insulator is very thick, $T$ is zero and there is no pair tunneling.

Let

$$\psi_1 = n_1 \text{e}^{i\theta_1} ; \quad \psi_2 = n_2 \text{e}^{i\theta_2} .$$

Then

$$\frac{\partial \psi_1}{\partial t} = \frac{1}{2} n_1 \text{e}^{i\theta_1} \frac{\partial n_1}{\partial t} + i \psi_1 \frac{\partial \theta_1}{\partial t} = -i T \psi_2 ,$$

(3)

with use of (1) in the form $\partial \psi_1 / \partial t = -i T \psi_2$. Similarly,

$$\frac{\partial \psi_2}{\partial t} = \frac{1}{2} n_2 \text{e}^{i\theta_2} \frac{\partial n_2}{\partial t} + i \psi_2 \frac{\partial \theta_2}{\partial t} = -i T \psi_1 .$$

(4)

We multiply (3) by $n_1 \text{e}^{-i\theta_1}$ to obtain, with $\delta \equiv \theta_2 - \theta_1$,

$$\frac{1}{2} \frac{\partial n_1}{\partial t} + i n_1 \frac{\partial \theta_1}{\partial t} = -i T (n_1 n_2) \text{e}^{i\delta} .$$

(5)

We multiply (4) by $n_2 \text{e}^{-i\theta_2}$ to obtain

$$\frac{1}{2} \frac{\partial n_2}{\partial t} + i n_2 \frac{\partial \theta_2}{\partial t} = -i T (n_1 n_2) \text{e}^{-i\delta} .$$

(6)
Figure 2 Current-voltage characteristic of a Josephson junction. DC currents flow under zero applied voltage up to a critical current $i_c$: this is the DC Josephson effect. At voltages above $V_c$ the junction has a finite resistance, but the current has an oscillatory component of frequency $\omega = 2eV/h$: this is the AC Josephson effect.

Now equate the real and imaginary parts of (5) and similarly of (6):

\[
\frac{\partial n_1}{\partial t} = 2T(n_1n_2)^{1\over 2} \sin\delta; \quad \frac{\partial n_2}{\partial t} = -2T(n_1n_2)^{1\over 2} \sin\delta; \quad (7)
\]
\[
\frac{\partial \theta_1}{\partial t} = -T(n_1)^{1\over 2} \cos\delta; \quad \frac{\partial \theta_2}{\partial t} = -T(n_2)^{1\over 2} \cos\delta. \quad (8)
\]

If $n_1 \equiv n_2$ as for identical superconductors 1 and 2, we have from (8) that

\[
\frac{\partial \theta_1}{\partial t} = -\frac{\partial \theta_2}{\partial t}; \quad \frac{\partial}{\partial t} (\theta_2 - \theta_1) = 0. \quad (8a)
\]

From (7) we see that

\[
\frac{\partial n_2}{\partial t} = -\frac{\partial n_1}{\partial t}. \quad (9)
\]

The current flow from (1) to (2) is proportional to $\partial n_2/\partial t$ or, the same thing, $-\partial n_1/\partial t$. We therefore conclude from (7) that the current $J$ of superconductor pairs across the junction depends on the phase difference $\delta$ as

\[
J = J_0 \sin \delta = J_0 \sin(\theta_2 - \theta_1), \quad (10)
\]

where $J_0$ is proportional to the transfer interaction $T$. The current $J_0$ is the maximum zero-voltage current that can be passed by the junction.

With no applied voltage a DC current will flow across the junction, (Fig. 2), with a value between $J_0$ and $-J_0$ according to the value of the phase difference $\theta_2 - \theta_1$. This is the DC Josephson effect.
AC JOSEPHSON EFFECT

Let a voltage \( V \) be applied across the junction. We can do this because the junction is an insulator. An electron pair experiences a potential energy difference \( qV \) on passing across the junction, where \( q = -2e \). We can say that a pair on one side is at potential energy \(-eV\) and a pair on the other side is at \( eV\). The equations of motion that replace (1) are

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2 - eV \psi_1 ; \quad \frac{i\hbar}{\partial t} \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1 + eV \psi_2 .
\] (11)

We proceed as above to find in place of (5) the equation

\[
\frac{1}{2} \frac{\partial n_1}{\partial t} + in_1 \frac{\partial \theta_1}{\partial t} = ieV n_1 \hbar^{-1} - iT(n_1 n_2)^{\frac{1}{2}}e^{i\delta} .
\] (12)

This equation breaks up into the real part

\[
\frac{\partial n_1}{\partial t} = 2T(n_1 n_2)\frac{1}{2} \sin \delta ,
\] (13)

exactly as without the voltage \( V \), and the imaginary part

\[
\frac{\partial \theta_1}{\partial t} = \frac{eV}{\hbar} - T\left(\frac{n_2}{n_1}\right)^{\frac{1}{2}} \cos \delta ,
\] (14)

which differs from (8) by the term \( eV/\hbar \).

Further, by extension of (6),

\[
\frac{1}{2} \frac{\partial n_2}{\partial t} + in_2 \frac{\partial \theta_2}{\partial t} = -i eV n_2 \hbar^{-1} - iT(n_1 n_2)^{\frac{1}{2}}e^{-i\delta} ,
\] (15)

whence

\[
\frac{\partial n_2}{\partial t} = -2T(n_1 n_2)\frac{1}{2} \sin \delta ;
\] (16)

\[
\frac{\partial \theta_2}{\partial t} = -\frac{eV}{\hbar} - T\left(\frac{n_1}{n_2}\right)^{\frac{1}{2}} \cos \delta .
\] (17)

From (14) and (17) with \( n_1 \equiv n_2 \), we have

\[
\frac{\partial (\theta_2 - \theta_1)}{\partial t} = \frac{\partial \delta}{\partial t} = -\frac{2eV}{\hbar} .
\] (18)

We see by integration of (18) that with a dc voltage across the junction the relative phase of the probability amplitudes vary as

\[
\delta(t) = \delta(0) - \frac{2eVt}{\hbar} .
\] (19)
The current is now given by (10) with (19) for the phase:

$$J = J_0 \sin \left[ \delta(0) - \frac{2eVt}{\hbar} \right].$$

(20)

The current oscillates with frequency

$$\omega = \frac{2eV}{\hbar}.$$  

(21)

This is the ac Josephson effect. A dc voltage of 1 μV produces a frequency of 483.6 MHz.

The relation (21) says that a photon of energy $\hbar \omega = 2eV$ is emitted or absorbed when an electron pair crosses the barrier. By measuring the voltage and the frequency it is possible to obtain a very precise value of $e/\hbar$.

MACROSCOPIC QUANTUM INTERFERENCE

We saw in Advanced Topic J that the phase difference $\theta_2 - \theta_1$ around a closed circuit which encompasses a total magnetic flux $\Phi$ is given by

$$\theta_2 - \theta_1 = \frac{2e}{\hbar c} \Phi.$$  

(22)

The flux is the sum of that due to external fields and that due to currents in the circuit itself. We consider two Josephson junctions in parallel, as in Fig. 3. No voltage is applied.

Let the phase difference between points 1 and 2 taken on a path through junction a be $\delta_a$. When taken on a path through junction b, the phase difference is $\delta_b$. In the absence of a magnetic field these two phases must be equal. Now let the flux $\Phi$ pass through the interior of the circuit. We can accomplish this

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Figure 4 Experimental trace of $J_{\text{max}}$ versus magnetic field showing interference and diffraction effects for two junctions A and B. The field periodicity is 39.5 and 16 mG for A and B, respectively. Approximate maximum currents are 1 mA (A) and 0.5 mA (B). The junction separation is 3mm and junction width 0.5 mm for both cases. The zero offset of A is due to a background magnetic field. [After R. C. Jaklevic, J. Lambe, J. E. Mercereau and A. H. Silver, Phys. Rev. 140, A1628 (1965).]

by a straight solenoid normal to the plane of the paper and lying inside the circuit. By (22)

$$\delta_b - \delta_a = \frac{2e}{\hbar c} \Phi \ .$$

or

$$\delta_b = \delta_0 + \frac{e}{\hbar c} \Phi \ ; \quad \delta_a = \delta_0 - \frac{e}{\hbar c} \Phi \ .$$

The total current is the sum of $J_a$ and $J_b$. The current through each junction is of the form (10), so that

$$J_{\text{total}} = J_0 \left\{ \sin \left( \delta_0 + \frac{e}{\hbar c} \Phi \right) + \sin \left( \delta_0 - \frac{e}{\hbar c} \Phi \right) \right\} = 2J_0 \sin \delta_0 \cos \frac{e\Phi}{\hbar c} \ .$$

The current varies with $\Phi$. The magnitude of the current has maxima when

$$\frac{e\Phi}{\hbar c} = r\pi \ , \quad r = \text{integer} \ .$$

The periodicity of the current is shown in Fig. 4. The short period variation is the interference effect of the two junctions predicted by (25) and (26). The longer variation is a diffraction effect and arises from the finite dimensions of each junction, which causes $\Phi$ to depend on the particular path we integrate over. The diffraction effect was responsible for the failure to observe pair tunneling in early experiments on single particle tunneling. The single particles are not liable to diffraction effects, but unless special care in construction and magnetic shielding is taken the contribution of the pairs is washed out by diffraction.
References

B. D. Josephson, Physics Letters 1, 251 (1962). (Difficult.)
R. Feynman, Lectures on physics, Addison-Wesley, 1963, Vol. 3, Sec. 21/9. (Simplest discussion.)

FOR AC JOSEPHSON EFFECT: