TRANSPORT PROCESSES IN PLASMA WITHOUT MAGNETIC FIELD

7.1 CHARGED PARTICLE DIRECTED MOTION AND ENERGY TRANSPORT IN WEAKLY IONIZED PLASMA

This chapter discusses the processes governing the balance of charged particles and their energies in a plasma without a magnetic field. The main processes here are the directed motion of the charged particles (particle transport), the directed transport of their energies, and the particle energy exchange in collision. In considering transport processes we use the moments equations obtained and analyzed in the preceding chapter. In most cases we assume stationary conditions and neglect terms proportional to the time derivatives. As before, we consider a three-component plasma consisting of electrons, single-charged positive ions, and neutral atoms. The transport processes in such a plasma depend on the degree of ionization. Two extremes can therefore be distinguished, namely, a weakly ionized plasma, in which the frequency of collisions of electrons and ions with atoms greatly exceeds that of collisions of these particles with one another,

\[ \frac{\delta q}{\delta t} = - \left( \frac{3}{4} \nu_{\text{ee}} + \frac{1}{4} \nu_{\text{ei}} \right) q_e + \frac{1}{2} n_e \nu_{\text{ee}} q_e + \frac{5}{8} \nu_{\text{ei}} (T_e - T_i) (u_e - u_i) n_e \]  

and a highly ionized plasma, which is described by the reverse inequalities. The degree of ionization characteristic of transition between the two cases is illustrated for a hydrogen plasma in Fig. 7.1.

We first consider the transport processes in a weakly ionized plasma. The directed motion of the charged particles is defined by the first-moment equations. When the viscosity effects are negligible (see Section 7.2), these equations reduce to the vector equality 6.62. In the absence of
Hereafter we neglect the terms of the moments equations proportional to the time derivatives. The condition of such neglect in Eq. 7.2 can easily be obtained by comparing the first term with the friction force (Eq. 7.4). This condition has the form

$$\left| \frac{m_\alpha \delta u_\alpha}{\delta t} \right| = \frac{m_\alpha u_\alpha}{\tau} \ll \left| R_{\alpha} \right| = \frac{m_\alpha u_\alpha v_{\alpha u}}{\tau}$$

or

$$v_{\alpha u} \tau \gg 1$$  \hspace{1cm} (7.5)

This means that the use of the stationary solution of Eq. 7.2 is legitimate when the characteristic time of variation in plasma parameters $\tau$ greatly exceeds the time between collisions $1/v$. Dropping the first term in Eq. 7.2 and substituting Eq. 7.4 for the collision term, we reduce Eq. 7.2 to

$$Z_\alpha e E - \frac{1}{n} \nabla(n T_\alpha) - \frac{1}{\mu_\alpha \nu_{\alpha u}} \frac{\nabla n}{\mu_\alpha \nu_{\alpha u}} = 0$$

(7.6)

The equality obtained enables us to find the directed velocity,

$$u_\alpha = \frac{Z_\alpha e E}{\mu_\alpha \nu_{\alpha u}} - \frac{T_\alpha}{\mu_\alpha \nu_{\alpha u}} \frac{\nabla n}{\mu_\alpha \nu_{\alpha u}}$$

(7.7)

which is a sum of three terms:

$$u_\alpha = u_{\alpha E} + u_{\alpha n} + u_{\alpha T}$$

(7.8)

The first of these, $u_{\alpha n}$, determines the directed velocity associated with the acceleration of the charged particles in an electric field. The proportionality factor between velocity and field $b_\alpha$ is called mobility,

$$u_{\alpha E} = Z_\alpha b_\alpha E, \quad b_\alpha = \frac{e}{\mu_\alpha \nu_{\alpha u}}$$

(7.9)

The second term describes diffusion due to the inhomogeneity of the charged particle density. The directed diffusive velocity is proportional to the relative density gradient

$$u_{\alpha n} = -D_\alpha \frac{\nabla n}{n}, \quad D_\alpha = \frac{T_\alpha}{\mu_\alpha \nu_{\alpha u}}$$

(7.10)

The proportionality factor $D_\alpha$ is referred to as the diffusion coefficient.

The last term in Eq. 7.8 describes the diffusion due to the temperature gradient—the so-called thermal diffusion. The directed thermal diffusion velocity $u_{\alpha T}$ can be represented in a form similar to Eq. 7.10:

$$u_{\alpha T} = -D_{\alpha T}^* \frac{\nabla T_\alpha}{T_\alpha}, \quad D_{\alpha T}^* = \frac{T_\alpha}{\mu_\alpha \nu_{\alpha u}}$$

(7.11)

where $D_{\alpha T}^*$ is the thermal diffusion coefficient.
TRANSPORT PROCESSES IN PLASMA

There exists a definite relationship between the coefficients $b$, $D$, and $D_T$, which determine the directed motion. In particular, using Eqs. 7.9 and 7.10 we obtain the relation between the diffusion and mobility coefficients

$$\frac{D_\alpha}{b_\alpha} = \frac{T_u}{e}$$

(7.12)

which is termed the Einstein relation. For particle velocity distributions close to the equilibrium one it is valid for any velocity dependence of particle collision frequency.

The total directed velocity of charged particles can be expressed via the above transport coefficients. Substituting Eqs. 7.9-7.11 into Eq. 7.8, we find for the electrons

$$\mathbf{u}_e = -b_e \mathbf{E} - D_e \nabla n - D_T \nabla T$$

(7.13)

where $b_e = e/m_e \nu_{ea}$, $D_e = T_e/m_e \nu_{ea}$, and for the ions

$$\mathbf{u}_i = b_i \mathbf{E} - D_i \nabla n - D_T \nabla T$$

(7.14)

where $b_i = 2e/m_i \nu_{ia}$, $D_i = 2T_i/m_i \nu_{ia}$. Using these expressions, we can also find the current density in the plasma, which is the sum of the electron and ion components:

$$\mathbf{j} = n_e \mathbf{u}_e - n_i \mathbf{u}_i = en(b_e + b_i) \mathbf{E} + e(D_e - D_i) \nabla n$$

$$+ en \left( D_T \frac{\nabla T}{T_e} - D_T \frac{\nabla T}{T_i} \right)$$

(7.15)

The ion components in this sum can usually be neglected, since the transport coefficients contained in it are inversely proportional to the masses. The first term in Eq. 7.15 determines the plasma conductivity $\sigma$ in a constant electric field:

$$J = \sigma E, \quad \sigma = enb_e = \frac{ne^2}{m_e \nu_{ea}}$$

(7.16)

One can see that it is proportional to the electron density.

We have obtained expressions describing the directed velocity of the charged particles when the frequency of their collisions with the neutral particles is velocity independent. Generally, with an arbitrary velocity dependence the expressions for the directed velocity components determined by the electric field and density and temperature gradients are similar to Eqs. 7.9-7.11. Here the transport coefficients contain certain collision frequencies averaged over velocity distributions; the frequencies differ by a factor of the order of unity in the various coefficients.

For a near-Maxwellian charged particle velocity distribution the directed velocity can be obtained by substituting the expression for the collision term derived in Section 6.3 into the first-moment equation. The expression consists of two terms, the friction force and the thermal force (see Eq. 6.58):

$$m_e \frac{\delta n_e}{\delta t} = \mathbf{R}_{ea} + \mathbf{R}^T_{ea}$$

(7.17)

As before, the friction force can be represented in the form 7.4 if we use the averaged collision frequency. In the first approximation the averaging law is given by the following equality (see Eq. 6.56):

$$\nu_{ea} = \frac{1}{3} \sqrt{\frac{2}{\pi}} \left( \frac{m_e}{T_{ea}} \right)^{\frac{3}{2}} \int_0^\infty w^3 \nu_{ea}(w) \exp \left( \frac{m_e w^2}{2T_{ea}} \right) dw$$

(7.18)

where the temperature, $T_{ea} = (m_e T_e + m_i T_i)/(m_e + m_i)$, corresponds to the relative-velocity distribution. In particular, if the collision frequency is proportional to the velocity $\nu_{ea} = \nu$ the averaged value of Eq. 7.18 is equal to

$$\nu = \frac{8}{3} \sqrt{\frac{2}{3\pi}} \nu_e^*(\nu_T)$$

and if the collision frequency is inversely proportional to the velocity it is equal to

$$\nu = 2 \sqrt{\frac{2}{3\pi}} \nu_e^*(\nu_T)$$

where $\nu_T = \sqrt{3T_{ea}/m_e}$.

A thermal force is induced by a temperature gradient and is equal, without a magnetic field, to

$$\mathbf{R}^T_{ea} = g_T \nabla T_{ea}$$

(7.19)

where the coefficient $g_T$ is determined by the velocity dependence of $\nu_{ea}$. The value of $g_T$ coincides, with an accuracy to a factor of the order of unity, with Eq. 6.60:

$$g_T = \frac{T_{ea}}{\nu_{ea} \partial T_{ea}}$$

To find a more accurate value of the effective collision frequency, which
determines the friction force, and to calculate the value of the coefficient 
$g_T$, which defines the thermal force, we have to solve the motion and 
heat flux equations simultaneously (see Section 6.3).

Substitution of the collision term 7.17 into Eq. 7.2 leads back to the 
former expressions for the velocity components (Eqs. 7.9-7.11). The 
coefficients of mobility ($b$) and diffusion ($D$) also satisfy the former 
expressions 7.9 and 7.10 if the collision frequency is replaced by its 
averaged value. Accordingly the relation between $D$ and $b$ is given by 
Eq. 7.12 as before. In determining the thermal diffusion coefficient we 
must consider the effect of the thermal force in Eq. 7.18. Adding it to the 
force related to the pressure gradient, we obtain

$$ D_T = \left( \frac{T_a}{\nu_{na} \nu_{ea} n} \right) (1 - g_T). $$(7.20)

This equation shows that the thermal diffusion coefficient may be either 
greater (at $g_T - \frac{dv}{dT} < 0$) or less (at $g_T - \frac{dv}{dT} > 0$) than the diffusion 
coefficient.

Let us now obtain heat fluxes that characterize the charged particle 
ergy transport. The equations for the heat flux vector $\mathbf{q}$ (third-moment 
equations) were given in Section 6.4. For our stationary case without a 
magnetic field they have the following form (see Eq. 6.80):

$$ \frac{5}{2} \frac{n}{m} \nabla T_a = \frac{\delta q_e}{\delta t} $$

In a weakly ionized plasma only collisions between charged and neutral 
particles are substantial. For electrons the corresponding collision term 
at a velocity-independent collision frequency is defined by Eq. 6.82: 
$\delta q_e/\delta t = -\nu_{ea} q_e$. Substituting it into Eq. 7.21, we obtain the expression 
for the electron heat flux

$$ q_e = -\frac{5}{2} \frac{nT_a}{m_e \nu_{ea}} \nabla T_a $$

(7.22)

The proportionality factor between heat flux and temperature gradient is 
referred to as the thermal conductivity coefficient, and its ratio to the 
density as the temperature conductivity coefficient. For the case at hand 
these coefficients are equal to, respectively,

$$ \kappa_e = \frac{5}{2} \frac{nT_a}{m_e \nu_{ea}}, $$

$$ \chi_e = \frac{\kappa_e}{n} = \frac{5}{2} \frac{T_a}{m_e \nu_{ea}} = \frac{5}{2} D_T. $$

(7.23)

Note that the temperature conductivity coefficient differs from the 
electron diffusion coefficient (Eq. 7.13) only by a factor of the order of 
unity. If the electron collision frequency is velocity dependent, the heat 
flux expression becomes more complicated. In addition to the term 
similar to Eq. 7.22 it contains a term proportional to the directed 
electron velocity. As a result $q_e$ takes the form

$$ q_e = -\frac{5}{2} \frac{g_e}{m_e \nu_{ea}} nT_a \nabla T_a + g_T nT_a u_e $$

(7.24)

The numerical values of the coefficients $g_e$ and $g_T$ are obtained from the 
dependence of $\nu_{ea}$ on $v$.

The heat flux of ions is often associated with that of the neutral atoms, 
since they effectively exchange energy on collision. For this reason 
the ion and atom heat flux equations often must be considered simultane­ously. Substituting the collision terms 6.87 and 6.88 into Eqs. 7.21 
for the ions and the neutral atoms (these terms are valid for velocity­
dependent collision frequencies $\nu_{ia}, \nu_{ia}$), neglecting the ion-ion col­
lisions, and taking into account that $\nu_{ia} = (n/n_{ia}) \nu_{ia}$ and $m_i = m_a$, we get

$$ \frac{5}{2} \frac{n}{m_i} \nabla T_i = \nu_{ia} \left[ -\frac{3}{4} q_i + \frac{1}{4} n_i \frac{q_i}{n_i} + \frac{5}{8} n_i (T_i - T_a) (u_i - u_a) \right], $$

$$ \frac{5}{2} \frac{n}{m_a} \nabla T_a = \nu_{ia} \left[ -\frac{3}{4} q_i + \frac{1}{4} n_i \frac{q_i}{n_i} + \frac{5}{8} n_i (T_i - T_a) (u_i - u_a) \right] - \frac{1}{4} \nu_{ea} q_a. $$

Solving these equations for $q_i$ and $q_a$, and taking into consideration that for 
a weakly ionized plasma $n_i \approx n_a$, we find

$$ q_i = -\frac{10}{3} \frac{n_i T_i}{m_i \nu_{ia}} \nabla T_i + \frac{5}{6} n_i (T_i - T_a) (u_i - u_a) $$

$$ -10 \frac{n_i T_a}{m_i \nu_{ia}} \nabla T_a; $$

$$ q_a = -10 \frac{n_i T_a}{m_a \nu_{ia}} \nabla T_a - \frac{10}{3} \frac{n_i T_i}{m_i \nu_{ia}} \nabla T_i $$

(7.25)

In the expression for the neutral atom heat flux the ion terms can be 
neglected if $n_i T_i^2 \approx n_a T_a^2$. This flux is controlled only by atom-atom 
collisions. In the expression for the ion heat flux at $T_a \approx T_i$, the last term
can be neglected, and then it takes a form similar to Eq. 7.24:

\[
q_i = -\frac{10}{3} n_i T_i \frac{\varepsilon}{m_e} \varepsilon \nabla T_i + \frac{5}{6} n_i T_i u_i
\]

(7.26)

The structure of the expression remains unchanged for a velocity-dependent collision frequency as well, but the numerical coefficients will change, of course.

### 7.2 ELECTRON MOBILITY, DIFFUSION, AND THERMAL CONDUCTIVITY COEFFICIENTS

The coefficients of mobility, diffusion, and thermal conductivity of electrons in a weakly ionized plasma can be obtained by a method using the expansion of the distribution function in the parameters characterizing its anisotropy. Such a method was described in Chapter 5. As demonstrated in Sections 5.1 and 5.2, this expansion is of a rapidly converging type. Restricting ourselves to the first two terms, we can represent the distribution function as in Eq. 5.20:

\[
f(v) = f_0(v) + \frac{\varepsilon}{v_i} f_1(v)
\]

(7.27)

The first term in the sum is the isotropic distribution function component depending solely on \(v\). The directed component appearing in the second term is responsible for the directed electron velocity. In accordance with Eq. 5.13, we find

\[
u = \int_0^\infty v f(v) dv = \frac{1}{3} \int_0^\infty v f_1(v) dv = \frac{4\pi}{3} \int_0^\infty v^3 f_1(v) dv
\]

(7.28)

Substituting Eq. 7.27 into the kinetic equation and making allowance for the smallness of the second term, we can easily obtain the equation for both components \(f_0\) and \(f_1\) (see Section 5.2). The equation defining \(f_1\) (see Eq. 5.22) has the following form for the stationary case \((\partial(n f)/\partial t = 0)\) without a magnetic field:

\[-\frac{Ne E}{m_e} \frac{df_0}{dv} + \varepsilon \nabla (n f_0) = S_t
\]

(7.29)

The collision term \(S_t\) was obtained in Section 5.3. Taking into account elastic and inelastic collisions, we can write it as

\[S_t = -n f_e \nu_e'(v)
\]

(7.30)

where \(\nu_e' = \nu' + \nu^l + \nu^h\) is the summary frequency of electron–atom collisions. Equation 7.29 makes it possible to find the function \(f_1\), due account being taken of Eq. 7.30:

\[
f_1 = -\frac{Ne E}{m_e} \frac{df_0}{dv} \frac{\varepsilon}{\nu_e} \nabla (n f_0) \frac{1}{n}
\]

(7.31)

Substituting Eq. 7.31 into Eq. 7.28, we obtain the directed electron velocity:

\[
u = 4\pi \frac{Ne E}{m_e} \int_0^\infty \frac{v^3}{\nu_e} df_0 dv - \frac{4\pi}{3} \int_0^\infty \frac{v^4}{\nu_e} f_0 dv
\]

(7.32)

(here the operator \(\partial\) is taken out of the integral). In conformity with the definition given in Section 7.1 (see Eqs. 7.8–7.11) it can be expressed via the mobility and diffusion coefficients:

\[
u = -b_e \frac{1}{n} \nabla (D_e n) = -b_e \frac{E}{n} - D_e \frac{\nabla n}{n} - D^T_e \frac{\nabla T_e}{T_e}
\]

(7.33)

where

\[
b_e = -\frac{4\pi \varepsilon}{3 m_e} \int_0^\infty \frac{v^3}{\nu_e} df_0 dv;
\]

\[
D_e = \frac{4\pi}{3} \int_0^\infty \frac{v^4}{\nu_e} f_0 dv;
\]

\[
D^T_e = T_e \frac{\partial D_e}{\partial T_e}
\]

(7.34–7.36)

Equations 7.34 and 7.35 for the coefficients \(b_e\) and \(D_e\) can be written in a form similar to Eqs. 7.9 and 7.10:

\[
b_e = \frac{-e}{m_e \nu_e} ; \quad D_e = \frac{T_e}{m_e \nu_e^2}
\]

(7.37)

if we appropriately determine the effective collision frequencies \(\nu_e^l\) and \(\nu_e^h\). Generally, with an arbitrary distribution function \(f_0\) these frequencies are different. Comparing Eqs. 7.34 and 7.35 with Eq. 7.37, we find for the collision frequency determining the mobility:

\[
\nu_e^l = \left[ \frac{4\pi}{3} \int_0^\infty \frac{v^3}{\nu_e} df_0 dv \right]^{-1} \left[ \frac{4\pi}{3} \int_0^\infty \frac{v^4}{\nu_e^2} f_0 dv \right]^{-1}
\]

(7.38)

and for the collision frequency appearing in the diffusion coefficient:

\[
\nu_e^p = T_e \left[ \frac{4\pi}{3} \int_0^\infty \frac{v^4}{\nu_e} f_0 dv \right]^{-1}
\]

(7.39)

It is easy to ascertain that when \(\nu_e^l\) is independent of \(v\), both these
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values are equal to \( v'_{es} \). With a Maxwellian electron velocity distribution

\[
f_d(v) = \left( \frac{m_e}{2\pi T_e} \right)^{3/2} \exp \left( -\frac{m_e v^2}{2T_e} \right)
\]

the collision frequencies \( \nu_{es} \) and \( \nu_{es}^D \) in the mobility and diffusion coefficients are also equal. This leads to the relationship 7.12 between \( b_e \) and \( D_e \). The expression for the effective collision frequency, which is derived by substituting the Maxwellian distribution into Eqs. 7.38 or 7.39, is of the form

\[
\nu_{es}(T_e) = \left[ \frac{1}{3} \sqrt{\frac{2}{\pi}} \left( \frac{m_e}{T_e} \right)^{3/2} \int_0^\infty \frac{v^4}{\nu_{es}(v)} \exp \left( -\frac{m_e v^2}{2T_e} \right) dv \right]^{-1}
\]

For instance, when the collision frequency is proportional to the velocity \( \nu_{es} \sim v \) the effective frequency is equal to

\[
\nu_{es}(T_e) = \frac{\sqrt{3\pi}}{2} \nu_{es}(v_T)
\]

and when the collision frequency is inversely proportional to the velocity \( \nu_{es} \sim 1/v \)

\[
\nu_{es}(T_e) = \frac{3}{2} \sqrt{\frac{3\pi}{2}} \nu_{es}(v_T)
\]

where \( v_T = \sqrt{3T_e/m_e} \). Note that with a weak dependence \( \nu(v) \) the difference of the effective collision frequency (Eq. 7.40) from the approximate expression 7.18 given in Section 7.1 is small. For \( \nu \sim v \) and \( \nu \sim 1/v \) this difference is about 13%.

The relationship between the thermal-diffusion and diffusion coefficients is given by Eq. 7.36. In accordance with it and using the determination of \( D_e \) via the effective collision frequency \( \nu_{es}^D \) (Eq. 7.37), we obtain

\[
D_e^T = T_e \frac{\partial}{\partial T_e} \left( \frac{T_e}{m_e \nu_{es}^D} \right) = \frac{T_e}{m_e \nu_{es}^D} (1 - g_{Te})
\]

where \( g_{Te} = (T_e/\nu_{es}^D) \partial \nu_{es}^D / \partial T_e \). This equation corresponds to Eq. 7.20 given in Section 7.1. It enables us to find the relationship between the coefficients \( D_e \) and \( D_e^T \). As mentioned previously, when the collision frequency is velocity independent \( D_e = D_e^T \) (see Eq. 7.11). With a collision frequency proportional to the velocity \( \nu_{es} \sim v \), Eq. 7.41 results in the relation \( D_e^T = 1D \) and with \( \nu_{es} \sim 1/v \), in the relation \( D_e^T = 1D \).

Let us now obtain the electron heat flux. We first find the electron energy flux \( Q \sim \frac{1}{2} \nu_{es} (v^2) \) (Eq. 6.12) by expanding Eq. 7.27. It is clearly determined by the directed component of the distribution function

\[
Q_e = \frac{1}{6} n_{me} \int (v' \nu f(v) d'v = \frac{1}{6} n_{me} \int (v' \nu f(v) d'v
\]

\[
Q_e = \frac{4\pi}{6} n_{me} \int_0^\infty v' f(v) dv
\]

With low directed velocities the heat flux is related to the energy flux by the equation

\[
Q_e = \frac{1}{2} n_{me} [(v - u)(v - u') - \frac{1}{2} n_{me} (v^2) - u(v')]
\]

\[
-2(v(vu)) = Q_e - \frac{5}{2} n_{me} + u_e
\]

Substituting Eq. 7.42 for \( Q_e \) and Eq. 7.28 for \( u_e \) into Eq. 7.43, we get

\[
Q_e = \frac{2\pi}{3} n_{me} \int_0^\infty \left( v - \frac{5}{2} \nu_e \right) f(v) dv
\]

Using Eq. 7.31, we find

\[
Q_e = -4\pi \frac{n_{me} \nu_{es}^D}{m_e \nu_{es}^D} \left( v^2 - 5T_e/m_e \right) f_0 dv - \frac{4\pi}{6} m_e
\]

\[
\times \text{grad} \left[ \int_0^\infty \left( v^2 - 5T_e/m_e \right) f_0 dv \right] - \frac{10\pi}{3} (n_{me} + T_e) \int_0^\infty \frac{v^2}{\nu_{es}^D} f_0 dv\]

This expression can be represented as

\[
Q_e = -\frac{5}{2} n_{me} \frac{T_e}{m_e \nu_{es}^D} \text{grad} T_e - g_{me} \frac{n_{me} T_e}{m_e \nu_{es}^D} + g_{ee} \frac{T_e^2}{m_e \nu_{es}^D} \text{grad} \left( \frac{T_e^2}{m_e \nu_{es}^D} \right)
\]

where \( \nu_{es}^D \) is the collision frequency determining the diffusion coefficient (Eq. 7.38):

\[
\frac{g_{ee}}{\nu_{es}^D} = \frac{4\pi}{6} \frac{m_e}{T_e} \int_0^\infty \left( v^2 - 5T_e/m_e \right) f_0 dv \left( \frac{T_e}{m_e \nu_{es}^D} \right)
\]

\[
\frac{g_{me}}{\nu_{es}^D} = -\frac{4\pi}{6} \left( \frac{m_e}{T_e} \right) \int_0^\infty \left( v^2 - 5T_e/m_e \right) f_0 dv \left( \frac{T_e}{m_e \nu_{es}^D} \right)
\]

With the Maxwellian distribution function \( f_0 \) we find

\[
\frac{g_{ee}}{\nu_{es}^D} = \frac{g_{ee}}{\nu_{es}^D} = \frac{1}{6} \frac{1}{\sqrt{2\pi}} \int_0^\infty \left( x^2 - 5T_e/m_e \right) \exp \left( -\frac{x^2}{2} \right) dx
\]

\[
= -\frac{1}{6} \frac{1}{\sqrt{2\pi}} \int_0^\infty \left( \frac{1}{\nu_{es}^D} \right) x^4 \exp \left( -\frac{x^2}{2} \right) dx
\]
Comparison of this equation with Eq. 7.40, makes it easy to see that
\[ g_a = -T_e \frac{\partial v_{ea}}{\partial T_e} \frac{\partial \ln g_a}{\partial T_e} \]
and hence
\[ g_a = \frac{T_e}{v_{ea}} \frac{\partial v_{ea}}{\partial T_e} \] (7.47)

Bearing in mind that with the Maxwellian distribution \( g_{a1} = g_{a2} = g_a \), we transform Eq. 7.46 to
\[ q = -\frac{S}{2} \frac{T_{r}}{m_{V_{ea}}} \text{grad} T_e - g_a n T_e u \] (7.48)

The first term in Eq. 7.48 defines the electron thermal conductivity coefficient with an arbitrary dependence \( v'_{ea}(v) \), which is equal to
\[ \kappa_e = \frac{S}{2} \frac{T_{r}}{m_{V_{ea}}} (7.49) \]

The second term describes the heat transfer due to the directed motion. Note that the coefficient \( g_a \), which determines this transfer (Eq. 7.47), is equal to \( g_e \) appearing in the expression for the thermal-diffusion flux (see Eqs. 7.33 and 7.41). It can be shown that the equality of these coefficients at a near-Maxwellian velocity distribution results from the principle of symmetry of the kinetic coefficient (Onsager principle), which is well known in thermodynamics.

7.3 MECHANISM OF TRANSFER PROCESSES

Let us now consider the physical picture of the transfer of charged particles and of their energy under the effect of the electric field, the density gradients, and the temperature.

The charged particle motion in the electric field is a superposition of the random, Brownian motion and the directed motion due to acceleration by the field. When the directed velocity is, on the average, much less than the random [it is shown in Section 5.1 that such conditions are fulfilled for ions in a weak field (Eq. 5.5), and for electrons in any field], the motion pattern differs little from Brownian. The total velocity of each particle consists of the velocity of the random (thermal) motion \( w \) and the velocity under the effect of the electric field \( v_e \):
\[ v = w + v_e \] (7.50)

For simplicity we assume that collisions of charged particles with atoms lead to complete isotropy of velocities; their average velocity immediately upon collision is zero. Then for each particle that experienced a collision at an instant \( t_0 \) we can assume \( v_e(t_0) = 0 \), \( v(t_0) = w \). In the interval between collisions a charged particle in the electric field undergoes acceleration. Its velocity at instant \( t \) is equal to \[ v = \frac{f_e}{m} \frac{w}{v_e} \] on the assumption that the field strength \( E \) changes but slightly between collisions. Averaging the velocity over the particle group and taking into account that the average random velocity is zero, while the average time from the last collision is of the order of the reciprocal collision frequency, we obtain the equation for the directed velocity
\[ u_e = \langle v_e \rangle = \frac{Z e E}{mv} \] (7.51)

which corresponds to Eq. 7.7. Here \( v \) is the collision frequency averaged over collisions and velocities (naturally, the averaging law remains uncertain in a qualitative consideration). It can be seen that the obtained dependence of the directed velocity on the collision frequency \( (u_e - 1/v) \) is due to the fact that the collisions limit the time of acceleration in the electric field.

Let us now take a look at the diffusion mechanism. The diffusion results from the random motion of the particles. In a homogeneous plasma, in the absence of an electric field the heat flux of the particles across any area is offset by the reverse flux. With a density gradient, there appears an uncompensated particle flux in the direction opposite to the gradient. This flux is caused by the different particle density on the two sides of the area perpendicular to the gradient. Assuming that the density gradient is directed along the \( x \) axis, we compute the flux across an area perpendicular to this axis. We assume, as before, that the collisions result in complete randomization of the velocities. After a collision the velocity of a particle in the direction of the isolated area is equal to \( w_e = w \cos \theta \), where \( w \) is the random velocity and \( \theta \) is the angle between the velocity vector and the \( x \) axis. The density of the flux of particles of a given velocity and direction across the area is equal to
\[ d\Gamma_x = n w \cos \Theta f(w) d^3 w \] (7.52)

where \( n \) is the density of the particles in the area where they experienced the last collision, and \( f(w) d^3 w \) is the fraction of particles with the given velocity value and direction. The flux \( d\Gamma_x \) is controlled by collisions at a point with a coordinate \( x = x - l \cos \theta \), where \( l \) is the mean free path from the point of collision to the area. Assuming that the particle density changes only slightly over the length \( l \), we find
\[ d\Gamma_x = n (x - l \cos \Theta) w \Omega f(w) d^3 w \]
\[ = \frac{d\Gamma_x}{d x} = n w \cos \Theta \left( \frac{d w}{d x} \right) \Omega f(w) d^3 w \]
Averaging this expression over the directions, summing over the values of the random velocity, and taking into account that \(\langle \cos \theta \rangle = 0\) and \(\langle \cos^2 \theta \rangle = \frac{1}{2}\), we get

\[
\Gamma_x = nu_x = -\frac{1}{2} \langle wl \rangle \frac{dn}{dx} \quad (7.53)
\]

Equation 7.53 enables us to find the diffusion coefficient. Assuming that the mean path to the last collision is determined by the collision frequency \(\langle l \rangle \approx \lambda = w/\nu\), we have

\[
D = \frac{1}{3} \langle wl \rangle \approx \frac{1}{3} \left( \frac{w^3}{\nu} \right) = \frac{T}{mv}
\]

which agrees with Eq. 7.10. The dependence of the diffusion flux on the temperature and collision frequency is due to the nature of the diffusion transfer caused by the thermal motion of the particles; it is proportional to the thermal velocity and to the difference of densities over the mean free path.

The diffusion due to the gradient of the temperature of the charged particles can be considered in a similar way. In this case the difference of the flux along and against the gradient is due to the difference in the average values of the random particle velocity and the mean free path on the different sides of the isolated area perpendicular to the temperature gradient. Making use of Eq. 7.52 for the flux density of particles with a given value and direction of random velocity and assuming that the density of the random distribution of the particles is determined by the site of the last collision, we get

\[
d\Gamma_x = nw \cos \Theta f(w, x - l \cos \Theta) d^3w
\]

Averaging this equation over the angles \(\Theta\) and summing over the random velocities, we can find the density of the flux associated with the temperature gradient:

\[
\Gamma_x = n\nu_T = -\frac{n}{3} \left[ \int_{\omega} \langle w l \rangle \frac{df}{dx} d^3w \right]
\]

\[
= -\frac{1}{3} n \frac{\partial}{\partial x} \int_{\omega} \langle w l \rangle d^3w = -n \frac{\partial}{\partial x} \frac{1}{3} \langle wl \rangle
\]

and further, assuming \(l = w/\nu\), we obtain

\[
u_T = \frac{\partial}{\partial x} \left( \frac{T}{mv} \right) = -\frac{T}{mv} \left( 1 - \frac{T \nu}{mv} \right) \text{grad} T
\]

(7.55)

This equation for the directed velocity is similar to Eq. 7.11 (for \(\nu = \text{const}\)) and Eq. 7.20 (for the general case). As seen from Eq. 7.54, at a constant (velocity-independent) mean free path the thermal diffusion flux is determined by the difference of the thermal velocities over this path, which leads to the decomposition of the particle fluxes along and against the temperature gradient. The additional effect is due to the change of the mean free path itself along the gradient.

The energy transfer under the effect of the temperature gradient is also determined by the thermal motion of the particles. It also exists when no particle flux is observed (for instance, in a reference system where the directed velocity is zero). Indeed, particles with a higher average energy must come from a region with a higher temperature to a region with a lower temperature. Therefore, even if the forward and reverse particle fluxes are equal, there must exist an uncompensated flux of their energy. Let us find this flux. We assume, as before, that the temperature gradient is directed along the \(x\) axis. The density of the energy flux carried over by particles with a given velocity is obtained by multiplying the flux density of the particles (Eq. 7.52) by their energy:

\[
dQ_x = \frac{mw^2}{2} nw \cos \Theta f(w) d^3w
\]

Assuming, as before, that the velocity distribution of the particles is determined by the site of their last collision and that it changes only slightly over the distance between collisions, we find

\[
dQ_x = \frac{nmw^3}{2} \cos \Theta f(w, x - l \cos \Theta) d^3w
\]

\[
= \frac{nmw^3}{2} \cos \Theta \left( f - l \cos \Theta \frac{df}{dx} \right) d^3w
\]

where \(l = w/\nu\). Averaging the flux over the angles \(\Theta\) and summing over the velocities, we obtain

\[
Q_x = \frac{1}{6} nm \left[ \int_{\omega} \frac{w^4 df}{dx} d^3w = \frac{1}{6} nm \frac{\partial}{\partial x} \left( \int_{\omega} \frac{w^4}{\nu} f d^3w \right) \right]
\]

(7.56)

In particular, with a Maxwellian velocity distribution for the case where the collision frequency is velocity independent

\[
Q_x = \frac{1}{6} nm \frac{\partial}{\partial x} \left[ 4\pi \left( \frac{m}{2\pi T} \right)^{3/2} \right] \times \int_0^\infty w^4 \exp \left( \frac{mw^2}{2T} \right) dw = \frac{1}{2} \frac{2\pi T \partial T}{mv} \frac{\partial}{\partial x}
\]
To find the heat flux density $q_x$, we must deduct from $Q_x$ the energy flux associated with the directed motion:

$$q_x = \frac{1}{2} nm \langle (v_x - u_x)(v - u)^2 \rangle = Q_x - \frac{5}{2} nu_x T$$

Using Eq. 7.54, we find, for $v = \text{const}$,

$$q_x = \frac{5}{2} nT \frac{\partial T}{\partial x}$$ (7.57)

The expression 7.57 yields the thermal conductivity coefficient, whose value coincides with the product of the diffusion coefficient by the average particle energy, with an accuracy to a numerical factor. This reflects the analogy in the mechanisms of diffusion transfer of particles and energy transfer, both related to the random motion of the particles.

Note that the electric field and the concentration gradient may lead to an additional energy transfer. The expressions for the corresponding fluxes, which are similar to Eqs. 7.24 and 7.45, are readily obtained from consideration of the difference in the energy fluxes along and against the gradients. The difference in such fluxes in a reference system where the directed velocity is zero is associated with the dependence of the collision frequency on the particle energy. Owing to this dependence the time of acceleration in the electric field and the concentration gradient over the mean free path are different for particles of different energy. As a result, in a reference system where the particle fluxes along and against the gradients are compensated the energy fluxes are also different.

In conclusion we estimate the components of the viscous stress tensor, which determines the anisotropic part of the momentum flux (see Section 6.1). Let us estimate, for instance, the transfer of the $y$ component of the momentum along the $x$ axis associated with the gradient of the directed velocity $\partial u_y/\partial x$ and described by the component $\pi_{yx}$ of the viscosity tensor. The momentum flux arises because in the presence of the directed velocity gradient $\partial u_y/\partial x$ the momenta $mu_y$ transferred as a result of the thermal motion along the $x$ axis in both directions, are uncompensated. This flux is determined by the product of the average momentum at the site of the last collision and the particle flux. For a group of particles with a given thermal velocity we find

$$d\pi_{yx} = mu_y(x - l \cos \Theta) d\Gamma_x = nm \left( u_y - \frac{\partial u_y}{\partial x} \cos \Theta \right) w \cos \Theta f(w) d'w$$

Summing over the velocities, we get

$$\pi_{yx} = \frac{1}{2} nm(1w) \frac{\partial u_y}{\partial x} = -nT \frac{\partial u_y}{\partial x}$$ (7.58)

Similar relations are derived for the other tensor components. One can show that the tensor is generally defined by the relation

$$\pi_{ij} = -\eta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \text{div} \mathbf{u} \right)$$ (7.59)

where the coefficient $\eta = nT/\nu$ is called the viscosity coefficient.

To establish the conditions for neglecting the viscosity effects in the transfer equations we compare the components of the viscosity tensor, which characterizes the anisotropic part of the total pressure, with the scalar pressure. Then the corresponding criterion takes the form $|\pi_{\mu\nu}| = (nT/\nu)u/L < p = nT$, or $u < vL = \langle w \rangle L/\lambda$, where $L$ is the characteristic scale of directed-velocity change. Since the conditions $\lambda \ll L, u \ll \langle w \rangle$ are necessary for the entire consideration this inequality is fulfilled a priori; hence the viscosity effect on the transfer processes must be small.

7.4 AMBIPOLAR DIFFUSION

In the preceding sections we obtained expressions for the directed velocity of charged particles under the effect of the electric fields, the density gradients, and the temperature. The mobility and diffusion coefficients appearing in these expressions are inversely proportional to the mass; they are much larger for the electrons than for the ions. But in view of the quasi-neutrality condition, independent motion of the electrons and ions in a plasma is impossible. A rapid removal of electrons from some element of the plasma volume will inevitably give rise to an electric field, which will prevent their removal from this volume and will speed up the removal of ions.

Let us see, for instance, how charged particles diffuse in a long cylindrical tube, assuming that the principal mechanism of their removal is recombination on the tube walls. A typical radial distribution of the density of charged particles in a volume is given in Fig. 7.2. With this distribution the diffusion occurs from the axis to the walls (against the density gradient). Suppose that at some initial moment the quasi-neutrality condition is rigorously fulfilled throughout the volume. Then, in the subsequent period the electron diffusion flux greatly exceeds the ion flux (because $D_e \gg D_i$). As a result the walls become negatively charged and an excess positive charge accumulates in the volume. The charge separation leads to the formation of a radial electric field, which speeds the ions toward the walls and retards the electrons. The field must increase until the electron and ion fluxes become equal. The space charge then no longer changes; that is, a quasi-stationary state sets in. This diffusion regime is called ambipolar. For the ambipolar motion to
be maintained, the negative potential of the plasma boundaries (walls) must become high enough to considerably decrease the electron flux. The corresponding potential energy obviously must exceed the average thermal energy of the electrons; that is, $e(\varphi_0 - \varphi) > T_e$. The space charge that causes such a potential difference is ensured by the stationary difference in the electron and ion densities ($\Delta n = n_i - n_e$). In the plasma, the value of $\Delta n/n$ is low ($\Delta n \ll n$) throughout the volume, with the exception of the wall layers. Naturally, the presence of the walls is by no means necessary for the ambipolar diffusion. The charge separation required for the ambipolar regime is the result of plasma inhomogeneity due to any reasons.

Let us now describe quantitatively the ambipolar diffusion in a plasma containing electrons and single-charged ions. The quasi-neutrality condition for such a plasma amounts to the equality of the electron and ion densities throughout the volume, with the exception of the wall layers, whose thickness is of the order of the Debye length,

$$|n_e - n_i| \ll n_e, \quad n_e \approx n_i \approx n \quad (7.60)$$

To maintain quasi-neutrality, the changes in electron and ion concentrations in each volume element must be equal, that is, $\delta n_e/\delta t = \delta n_i/\delta t$ or, in conformity with the particle balance equations (Eq. 6.36),

$$-\text{div}(n_e u_e) + \frac{\delta n_e}{\delta t} = -\text{div}(n_i u_i) + \frac{\delta n_i}{\delta t}$$

The volume processes of appearance and removal of particles (ionization and recombination) in a three-component plasma cannot lead to a disturbance of quasi-neutrality—an electron and an ion appear or disappear in each of these processes simultaneously. Accordingly, $\delta n_e/\delta t = \delta n_i/\delta t$ and the condition for maintaining quasi-neutrality amounts to the equality of the flux divergences

$$\text{div}(n_e u_e) = \text{div}(n_i u_i) \quad (7.61)$$

This equation is equivalent to the expression $n u_i = n u_e + \Gamma_i$, where $\text{div} \Gamma_i = 0$, that is, the flux $\Gamma_i$ does not affect the density. It evidently determines the current in the plasma $j = e(n_e u_e - n_i u_i) = \epsilon \Gamma_i$. One can assume that this current is associated with the electric field induced by external sources. In the unidimensional case (when the flux $n u$ depends on a single coordinate) this is obvious, since the current can close only through external electrodes. Thus, in the absence of an external field, when there is no current in the plasma either and $\Gamma_i = 0$, the condition 7.61 leads to the equality of the directed velocities of the electrons and ions

$$u_e = u_i \quad (7.62)$$

The directed motion defined by Eq. 7.62 is actually the ambipolar diffusion.

In the presence of an external field the directed electron and ion velocity can be represented as a sum of the ambipolar velocity and the current velocity related to the field of the external sources $E_0$ via the mobility coefficients:

$$u_e = u_A - b_e E_0; \quad u_i = u_A + b_i E_0 \quad (7.63)$$

The condition 7.61 results in the equality

$$\text{div}[n(b_e + b_i)E_0] = 0$$

which yields the distribution of the field $E_0$ in the plasma.

Let us find the characteristics of the ambipolar motion. To this end we use Eqs. 7.13 and 7.14 for the directed velocity. When $|\text{grad} T/T| \ll |\text{grad} n/n|$ and the thermal diffusion is insignificant the directed electron and ion velocity is equal to

$$u_e = -D_e \frac{\text{grad} n}{n} - b_e E; \quad u_i = -D_i \frac{\text{grad} n}{n} + b_i E \quad (7.64)$$

By equating these velocities in conformity with Eq. 7.62 it is easy to find the ambipolar electric field, which forms automatically in the plasma to equalize the fluxes of oppositely charged particles:

$$E_A = \frac{D_e - D_i}{b_e + b_i} \frac{\text{grad} n}{n} \quad (7.65)$$
Considering that $D_i \gg D_e$ and $b_e \gg b_i$ and taking advantage of the relationship between $D_i$ and $b_i$ (Eq. 7.12), we find the approximate expression for $E_A$:

$$E_A \approx -\frac{D_i}{D_e} \frac{\text{grad} n}{n} = \frac{T_i}{e} \frac{\text{grad} n}{n}$$  \hspace{1cm} (7.66)

The electric field is directed oppositely to the density gradient. Therefore, as would be expected, it hinders the diffusion of the electrons and increases the ion flux. Since $E = -\text{grad} \phi$, we obtain from Eq. 7.66 the potential distribution

$$\phi - \phi_0 = \frac{T_e}{e} \ln \frac{n}{n_0}$$  \hspace{1cm} (7.67)

This equation determines, in particular, the potential difference between the central region of the plasma and its boundaries. It is seen that at a low density at the boundary $n_0 \ll n_0$, this difference greatly exceeds $T/e$.

The distribution obtained $n = n_0 \exp [e(\phi - \phi_0)/T_e]$ corresponds to the Boltzmann equation (Eq. 4.18). The equilibrium Boltzmann distribution in an ambipolar electric field arises because the ambipolar field leads to a near-total reflection of the electron flux from the walls (as demonstrated in Section 4.1 the absence of a directed motion is the condition for the existence of the equilibrium distribution).

Knowing the electric field strength, we can find the directed particle velocity. Substituting Eq. 7.65 into Eq. 7.64, we obtain the velocity of the joint (ambipolar) directed motion of the charged particles under the effect of the density gradient:

$$u_e = u_i = -D_A \frac{\text{grad} n}{n};$$  \hspace{1cm} (7.68)

$$D_A = \frac{D_i b_i + D_e b_e}{b_i + b_e}$$  \hspace{1cm} (7.69)

The expression 7.68 formally coincides with the expression for the diffusion flux velocity. The coefficient $D_A$ is called the ambipolar diffusion coefficient. Since $D_i \gg D_e$, $b_e \gg b_i$, we obtain from Eq. 7.69 $D_e \approx D_i + b_i D_i/b$, and further, from Eqs. 7.10 and 7.12,

$$D_A \approx D_i \left(1 + \frac{T_e}{T_i}\right) = \frac{T_e + T_i}{\mu_e \nu_a}$$  \hspace{1cm} (7.70)

It follows that the ambipolar diffusion coefficient is much less than the coefficient of free (unipolar) electron diffusion and greater than the ion diffusion coefficient $D_i < D_A < D_e$. Thus an ambipolar electric field greatly reduces the directed electron velocity.

In deriving equations of the ambipolar directed velocity we assumed the absence of temperature gradients. It is easy to include the thermal diffusion in the same way as was done above. Equating the complete expressions for the electron and ion fluxes, that is, Eqs. 7.13 and 7.14, one can find the ambipolar flux with an allowance for the temperature gradients. Then the electric field strength will be equal to

$$E_A = \frac{D_e - D_i}{b_e + b_i} \frac{\text{grad} n}{n} + \frac{D_i}{b_e + b_i} \frac{\text{grad} T_e}{T_e}$$

or, since $b_i \ll b_e$, $D_i \ll D_e$:

$$E_A = \frac{D_e}{b_e + b_i} \frac{\text{grad} n}{n} + \frac{D_i}{b_e + b_i} \frac{\text{grad} T_e}{T_e}$$

$$\approx \frac{T_e}{e} \left[\frac{\text{grad} n}{n} + (1 - g_{Te}) \frac{\text{grad} T_e}{T_e}\right]$$

where the relation 7.9, 7.10, and 7.20 for $b_i$, $D_i$, and $D_e$ are used.

The expression for the ambipolar directed velocity takes the form

$$u_A = -D_A \frac{\text{grad} n}{n} - D_{A'\nu} \frac{\text{grad} T_e}{T_e} - D_{A''\nu} \frac{\text{grad} T_i}{T_i}$$

(7.71)

where the ambipolar diffusion and thermal-diffusion coefficients are equal to

$$D_{A'} = \frac{b_i D_i + b_e D_e}{b_i + b_e} \approx D_i \left(1 + \frac{T_e}{T_i}\right);$$

$$D_{A''\nu} = \frac{b_i}{b_e + b_i} \approx D_i (1 - g_{Te});$$

$$D_{A''\nu} = \frac{b_e}{b_e + b_i} \approx D_i (1 - g_{Te})$$

Here, in conformity with 7.11 and 7.20, $D_i = T_i/\mu_e \nu_a$; $g_{Te} = (T_{ne}/T_{en})dJ_{en}/dT_{en}$.

When the collision frequencies are velocity independent, $g_{Te} = g_{Te} = 0$.

In this case the expressions for the ambipolar velocity and the ambipolar field strength can be represented as

$$u_A = -\frac{\text{grad}(D_A n)}{n} = -\frac{\text{grad}[n(T_e + T_i)]}{n \mu_e \nu_a}; \quad E_A = \frac{\text{grad}(n T_e)}{en}$$

(7.74)
7.5 CHARGED PARTICLE AND ENERGY BALANCE EQUATIONS FOR WEAKLY IONIZED PLASMAS

The expressions obtained for the directed velocity and the heat flux of charged particles make it possible to finally establish the particle and energy balance equations considered in Chapter 6. The stationary values of the transfer coefficients can be used if the characteristic times of change in the concentration and temperature of the charged particles greatly exceed the collision frequencies (see Eq. 7.5). We begin with the particle balance equation (see Chapter 6):

\[
\frac{\partial n}{\partial t} + \text{div}(n\mathbf{u}) = \frac{\delta n}{\delta t} \tag{7.75}
\]

As demonstrated in the preceding sections, the change in concentration is determined by the ambipolar component of the directed velocity. In the general case it is given by Eq. 7.72. When the frequencies of collisions of electrons and ions with atoms are velocity independent, we obtain, by substituting Eq. 7.74 into Eq. 7.75,

\[
\frac{\partial n}{\partial t} - \nabla(D_n \n) = \frac{\delta n}{\delta t} \tag{7.76}
\]

where \( D_n = (T_e + T_i)^{\mu_{ne} \mu_{ni}} \). This equation includes the density and temperature as unknowns. Therefore it must be solved simultaneously with the energy balance equations. The relative temperature gradients, however, are usually much lower than the relative gradients of charged particle density. In this case the directed velocity is defined by Eq. 7.68, and the charged particle balance equation 7.75 takes the form

\[
\frac{\partial n}{\partial t} - D_n \nabla n = \frac{\delta n}{\delta t} \tag{7.77}
\]

It is an equation in partial derivatives for the density and can be solved independently of the energy balance equations. To solve Eq. 7.77 one must know the initial density distribution and the boundary conditions.

Let us dwell on the latter. More often than not, these conditions result from the removal of charged particles to the dielectric or metal walls enclosing the plasma. In the wall regions one can isolate layers in which the quasi-neutrality conditions are not fulfilled (Fig. 7.3). Their thickness \((\Delta r)\) is of the order of the Debye radius; it is usually much less than the mean free path. Since on ambipolar removal the walls acquire a negative charge, the electric field in the layer is directed toward the wall. The flux of charged particles toward the wall is evidently determined by their density near the boundary layer \( n_{\text{w}} \), the average velocity in the direction of the layer \( v_{\text{w}} \), and the coefficient of particle reflection from the layer \( \eta_{\text{w}} \):

\[
\Gamma_{\text{w}} = (1 - \eta_{\text{w}}) n_{\text{w}} v_{\text{w}} \tag{7.78}
\]

The average velocity of the wall-bound ions at \( T_e > T_i \) may exceed their random velocity because of their acceleration in the ambipolar electric field (Eq. 7.66). At distances of several mean free paths from the wall this field sets up a potential difference of the order of \( T_e/e \); accordingly, near the layer boundary the ions accelerate to an energy of the order of \( T_e \), and their velocity \( v_{\text{w}} \approx (T_e/m_i)^{1/2} \). In the electric field of the layer, the ions move toward the wall with acceleration, and since the collisions in the layer are insignificant, practically all of them reach the wall. Besides, most of the ions reflected from the wall are returned by the field of the layer. Therefore the coefficient of reflection of the ions from the layer is low, \( \eta_{\text{w}} \ll 1 \). Bearing this in mind, we estimate the ion flux toward the wall at \( T_e > T_i \):

\[
\Gamma_{\text{i}} = (1 - \eta_{\text{i}}) n_{\text{i}} v_{\text{i}} = n_{\text{i}} \sqrt{T_i/m_i} \tag{7.79}
\]

When estimating the flux of electrons toward the wall one must remember that their average random velocity always greatly exceeds the directed velocity. Accordingly, their average velocity in the direction of the wall is determined by the random motion \( v_{\text{e}} = \frac{1}{2}(v_0) = \sqrt{T_e/m_e} \), and the electron flux toward the wall is equal to \( \Gamma_{\text{e}} = (1 - \eta_{\text{e}}) n_{\text{e}} \sqrt{T_e/m_e} \). In the absence of a current toward the wall the electron flux must be equal to that of the ions. This equality is possible only when the reflection coefficient of the electrons \( \eta_{\text{e}} \) is close to unity: \( 1 - \eta_{\text{e}} \approx \sqrt{m_i/m_e} \). It is easy to estimate the potential difference in the layer \( \Delta \phi \) ensuring the necessary reflection coefficient. Obviously, only those electrons whose energy of motion toward the wall exceeds \( e\Delta \phi \) can reach the wall through the layer. The flux of such electrons toward the layer boundary...
is equal to
\[ \Gamma_{ce} = n_e \int_{0}^{\infty} w_f e_a(w_e) \, dw_e \]
where the x axis is directed along the normal to the layer. With a
Maxwellian distribution \( f_e = (m_e/(2\pi T_e)^{3/2}) \exp(-m_e w_e^2/2T_e) \) we obtain
\[ \Gamma_{ce} = \frac{1}{\sqrt{2\pi}} n_e \sqrt{\frac{T_e}{m_e}} \exp\left(-\frac{e\Delta \phi_i}{T_e}\right) \quad (7.80) \]
Equating this flux to that of the ions (Eq. 7.79), we find the potential
difference ensuring ambipolar removal of particles from the plasma:
\[ \Delta \phi_i \approx \frac{T_e}{e} \ln \sqrt{m_j/m_e} \approx (4-7) \frac{T_e}{e} \quad (7.81) \]
To estimate the density of charged particles at the plasma/layer
boundary (in the quasi-neutrality region) it is necessary to equate the
charged particle flux at the layer boundary to the diffusion flux of the
particles from the plasma. The flux from the plasma can be determined
with the aid of equations for the ambipolar diffusion rate (Eq. 7.68)
\[ \Gamma_{ce} = D_A |\nabla n| = T_e \frac{n_0}{m_e n_e} \frac{\lambda_{ce}}{L} \]
where \( n_0 \) is the charged particle density in the central region of the
plasma and \( L \) is the characteristic dimensions of the plasma. Equating
this flux to the ion flux toward the layer (Eq. 7.79), we obtain the ratio
between the densities of the charged particles at the plasma boundary
and in the center
\[ \frac{n_e}{n_0} = \frac{1}{1 + n_0} \sqrt{\frac{T_e}{m_i}} = \frac{\lambda_{ce}}{L} \sqrt{\frac{T_e}{T_i}} \quad (7.82) \]
In the region of conditions where the transport equations are applicable
the change in plasma parameters over the mean free path must be
small. Therefore it is usually possible to put \( \lambda_{ce} \ll L, n_e \ll n_0 \) and
approximately assume the boundary density to be equal to zero, \( n_e = 0 \).
The zero boundary conditions are commonly employed in solving diffusion
problems.
Now we pass over to the charged particle energy balance equations
obtained in Section 6.4 from the second-moment equations. For a
weakly ionized plasma, in which only collisions of charged particles with
neutrals are substantial, the electron and ion energy balance equations

6.77 can be written thus:
\[ \frac{\partial T_e}{\partial t} + u_e \nabla T_e + \frac{2}{3} \frac{T_e}{T_i} \div u_e + \frac{2}{3} \nabla q_e \]
\[ = -\kappa_e \nu_e (T_e - T_i) + \frac{2}{3} \frac{m_e \nu_e^2}{(m_e + m_i)} u_e^2 \quad (7.83) \]
The expressions obtained above for the directed velocity \( u_e \) and the heat
flux \( q_e \) must be substituted into these equations. As a result, we get
nonlinear partial differential equations. Simultaneous solution of the
electron and ion energy balance equations and the particle balance
equation makes it possible, in principle, to find the distribution of the
density and the electron and ion temperatures in the plasma volume. In
the general case this problem is very complicated, of course.
Consider now the electron energy balance equation, assuming that the
frequencies of collisions of electrons and ions with atoms are velocity
independent. As shown in Section 6.4, the directed electron velocity can
be represented as a sum of the ambipolar velocity (Eq. 7.74) and the
current velocity due to the external field (Eq. 7.63):
\[ u_e = -\frac{1}{n} \nabla (D_n n) - \rho_b E_0 \]
where \( D_e = (T_e + T_i) \mu_e, b_e = e m_e v_{\text{ne}} \), and the field distribution \( E_0 \) must
meet the condition \( \nabla / \nabla = 0 \). The electron heat flux is
determined by the equality 7.22:
\[ q_e = -\chi_e \nabla \nabla T_e = -\frac{\chi_e}{n_e} \nabla \nabla T_e \]
where \( \chi_e = \frac{4}{7} T_e m_e \nu_e/n_e = D_e \). When substituting these expressions into Eq.
7.83 we take into account that \( D_e \approx D_n \) and the ambipolar term in the
expression for \( u_e \) is usually much less than the current velocity \( D_n/L \ll
b_e E_0 \). Neglecting the corresponding small terms, we obtain
\[ \frac{\partial T_e}{\partial t} + \frac{5}{3n} \nabla (D_n n \nabla T_e) - \frac{2}{3} D_n \frac{\nabla n \nabla}{n} \]
\[ - \frac{2}{3} \frac{e D_n E_0 \nabla \nabla - \epsilon D_n E_0 \nabla \nabla T_e}{T_e} = -\kappa_e \nu_e (T_e - T_i) + \frac{2}{3} \frac{e^2 \epsilon E_0^2}{m_e \nu_e} \quad (7.84) \]
Here the average coefficient of energy transfer on collisions \( \kappa_e \) and the
collision frequency \( \nu_e \) satisfy Eqs. 6.44 and 6.76; that is, they include
both elastic and inelastic collisions of electrons with atoms. The non­
linear equation 7.84 is rather complicated but can often be simplified,
because some of the terms on the left-hand side are negligibly small.
Thus in a gas-discharge plasma in long cylindrical tubes the external field
is directed along the axis, and the gradients along the radius; therefore,
paths one can neglect all the terms on the left-hand side that define the sufficiently large ratios of the characteristic dimensions to the mean free energy transfer. Comparing them with the first term on the right-hand side, which describes the electron energy transfer on collisions, we get the conditions for such neglect:

\[ L_T^2; \quad L_T \approx \frac{\lambda_a^2}{\kappa_{ea}}, \quad L_T^2 \approx \left( \frac{m_e T_e}{m_i T_i} \right)^{1/2} \frac{\lambda_a \lambda_n}{\kappa_{ea}} \]  
(7.85)

where \( L_T = (\text{grad} T_i/T_e)^{-1}, \) \( L_n = (\text{grad} n/n)^{-1} \) are the characteristic scales of change in density and temperature. If these conditions hold, Eq. 7.84 takes the simplest form:

\[ \frac{\partial T_e}{\partial t} = -\kappa_{ea} v_{ea}(T_e - T_s) + \frac{2}{3} \frac{e^2 E_i^2}{m_e} \]  
(7.86)

It defines the local relationship of the electron temperature with the strength of the field induced by external sources.

To solve the electron energy balance equation, with due regard for energy transfer, one must, along with the boundary conditions for the concentration, determine the boundary conditions for the electron temperature by estimating the energy transferred by the electrons leaving the plasma. As has been shown above (see Fig. 7.3), the wall layer has a potential barrier for electrons \( e\Delta \phi \) (Eq. 7.81), which greatly reduces their flux toward the wall and thus ensures the ambipolarity of their transport. The electrons with an energy above \( e\Delta \phi \) are reflected from the layer and return to the plasma without a change in energy. The electrons with an energy below \( e\Delta \phi \) pass through the layer and move away to the walls. They carry part of their energy (remaining after slowing down in the layer) to the walls. The remainder is spent on maintaining the potential difference in the layer; it is carried to the walls by ions accelerating in the layer. Thus the electron energy flux toward the layer boundary is carried by fast particles with an energy exceeding \( e\Delta \phi \). Let us determine it for the Maxwellian electron velocity distribution.

Assuming that the \( x \) axis is parallel to the normal to the layer, we find

\[ q_{ea} = n_e \int w_s \frac{m_e w^2}{2} f_e(w) dw \]  

\[ = n_e \left( \frac{m}{2 \pi T_{ea}} \right)^{3/2} \int_{\epsilon_{ea} w_s}^{w_s} \frac{w_s}{w} dw_s \int_{-\infty}^{\infty} dw_i \int_{-\infty}^{\infty} dw_i \times \frac{m_e w^2}{2 e^{2 (\epsilon_{ea} w_s)^2}} \]  

where the quantities \( n_e, T_{ea}, \) and \( q_{ea} \) are found near the layer boundary, and integration is done over the velocities \( w_s \) at which the barrier can be penetrated \( (m_{w_s}^2/2 > e\Delta \phi) \), and over all the velocities \( w_i \) and \( w_\perp \). Integration yields

\[ q_{ea} = \frac{1}{\sqrt{2\pi}} \frac{n_e T_{ea}^{3/2}}{m_e^{1/2}} \frac{2 + e\Delta \phi}{T_{ea}} \exp \left( \frac{e\Delta \phi}{T_{ea}} \right) \]  
(7.87)

The ratio of the heat flux to the particle flux at the layer boundary (Eq. 7.80) is equal to the average energy carried away from the plasma by a single electron:

\[ \frac{\varepsilon_0}{T_e} \approx q_{ea} = T_{ea} \left( 2 + \frac{e\Delta \phi}{T_{ea}} \right) = T_{ea} \left( 2 + \ln \sqrt{m_i/m_e} \right) \]  
(7.88)

To obtain the boundary conditions the heat flux toward the layer boundary must be equated to the heat flux from the plasma. Taking into account Eqs. 7.22 and 7.68, we obtain

\[ \frac{5}{2} n_e D_e \left| \frac{\text{grad} T_e}{T_e} \right| = T_{ea} \left( 2 + \ln \sqrt{m_i/m_e} \right) \frac{1}{T_e} \]  

\[ = T_{ea} \left( 2 + \ln \sqrt{m_i/m_e} \right) D_e (|\text{grad} n|) \]  

From this equality we find the ratio of the temperature and density gradients at the boundary

\[ \left( \frac{\text{grad} T_e}{T_e} \right)_e = \left( 2 + \ln \sqrt{m_i/m_e} \right) \frac{D_e}{D_a} \left( \frac{\text{grad} n}{n} \right)_e \]  
(7.89)

It is seen that near the boundary (\( \text{grad} T_e/T_s \approx (\text{grad} n/n) \)), and we can approximately assign the boundary conditions for \( T_e \) assuming (\( \text{grad} T_e)_e = 0 \). This condition stems from the substantial difference between the coefficient of ambipolar diffusion \( D_a \) and the coefficient of thermal conductivity \( \chi_e = 1/D_a \). Since \( D_a \ll \chi_e \) the transport of the thermal energy of the electrons from the central part to the periphery of the plasma is much faster than the ambipolar transport of the electrons themselves. At the same time the energy transport to the walls is associated with the removal of electrons to the walls, and therefore it occurs at a rate only slightly exceeding that of the ambipolar particle transport. It is the faster exchange of electron energy inside the plasma volume that equalizes the electron temperature in the volume.

The ion energy balance equation can be obtained by substituting Eqs. 7.63 and 7.74 for the directed velocity and Eq. 7.25 for the heat flux into
Eq. 7.83. As a result we derive a still more intricate equation than the one for the electrons, since the ion heat flux is, generally speaking, related to that of the atoms. We do not consider the equation here, but indicate the conditions under which heat transfer is insignificant. They result from a comparison of the equation terms proportional to the density and temperature gradients with the term defining the transfer of the ion energy to the neutral atoms on collisions. It is easy to see that these conditions (for \( E_0 \perp \nabla n, \nabla T \)) amount to the inequality

\[
L \gg \sqrt{T_0/T_i} \lambda_n
\]  

(7.90)

If this holds, the ion energy balance equation takes the form

\[
\frac{\partial T_i}{\partial t} = -\frac{1}{n} \nu_{in} (T_i - T_e) + \frac{1}{n} \nu_{in} m_i u_i
\]  

(7.91)

where, in accordance with Eq. 7.63,

\[
u_i = b_i E_0 - \frac{\nabla (D_A n)}{n} - \frac{2}{m_i \nu_{in}} \left[ eE_0 \frac{1}{n} \nabla n (T_e + T_i) \right]
\]

(here we took into account that \( m_i = m_{ne}, \mu_{in} = \frac{1}{m_i} \)).

7.6 CHARGED PARTICLE AND ENERGY BALANCE IN STATIONARY GAS DISCHARGE PLASMAS

To illustrate the application of the equations obtained we consider the charged particle and energy balance in a plasma of a stationary gas discharge maintained by a longitudinal electric field in a long cylindrical chamber. If the length of the chamber greatly exceeds its diameter, we can assume that the plasma parameters are independent of the longitudinal coordinate; that is, the gradients are perpendicular to the axis. The external electric field in such a plasma must be practically homogeneous. According to the analysis carried out in Section 7.5, as a result of the ambipolar regime of charged particle removal to the walls the relative electron temperature gradient near the walls is much lower than the relative density gradient. Therefore when the electron heating is uniform throughout the cross section the electron temperature can also be assumed constant, as a first approximation. Owing to the intensive energy exchange between the ions and the atoms, the ion temperature is usually much lower than the electron temperature and also changes insignificantly over the cross section. The distribution of charged particle densities is then described accurately enough by Eq. 7.77. For the stationary case, assuming \( \partial n/\partial t = 0 \), we have

\[
D_A \nabla n + \frac{\partial n}{\partial t} = 0
\]  

(7.92)

Recall that in a cylindrically symmetric plasma the density depends exclusively on the radius. Then

\[
D_A \frac{1}{r} \frac{d}{dr} \left( r \frac{dn}{dr} \right) + \frac{\partial n}{\partial t} = 0
\]  

(7.93)

The collision term of the equation \( \partial n/\partial t \) determines the efficiency of the ionization and recombination processes in the volume. It may generally include direct ionization on electron–atom collisions, stepwise ionization, electron–ion recombination, and electron capture with subsequent ion–ion recombination. Let us first discuss the simplest situation, when the only volume process appreciably affecting the particle balance is direct ionization. Here \( \partial n/\partial t = \nu' n \), and the equation takes the form

\[
D_A \frac{1}{r} \frac{d}{dr} \left( r \frac{dn}{dr} \right) + \nu' n = 0
\]  

(7.93)

where \( \nu' = (n, \nu' r, \nu) \) is the average ionization frequency defined by the electron energy distribution function. If the distribution function of the electrons is independent of their density Eq. 7.93 is found to be linear. Its particular solution, which is finite at \( r = 0 \), is known to be a zero-order Bessel function. It can be written as

\[
n = n_0 J_0 \left( \frac{r}{\Lambda} \right); \quad \Lambda = \sqrt{D_A \nu'}
\]  

(7.94)

For the density to vanish at the boundary (at \( r = a \)) the argument of the Bessel function at this point must be its root. The number of roots of the Bessel function is infinite, but it is only the solution corresponding to the first root (\( \zeta = 2.405 \)) that has a physical meaning, since only this solution is positive throughout the region \( r < a \). Bearing this in mind, we find

\[
\Lambda = \frac{a}{2.405}
\]  

(7.95)

The relation 7.95 determines the value of \( \Lambda \) and hence the density distribution (Eq. 7.94). This distribution, which is called the diffusion distribution, is depicted in Fig. 7.4 (curve 1). The characteristic length \( \Lambda \) is called the diffusion length. It defines the ratio between \( \nu' \) and \( D_A \).
In accordance with Eqs. 7.94 and 7.95 we find

$$v' = \frac{D_A}{\Lambda^2} \approx \frac{5.8 D_A}{a^2}$$  \hspace{1cm} (7.96)

This ratio is the condition for the equilibrium of density, that is, for the equality of the rates of production of charged particles and their diffusion removal from the plasma volume. The right-hand side of Eq. 7.96 is sometimes called the diffusion frequency of removal \(v_D\), and its reciprocal, the diffusion time \(\tau_D\):

$$\tau_D = \frac{1}{v_D} = \frac{\Lambda^2}{D_A}$$  \hspace{1cm} (7.97)

When solving the charged particle balance equation 7.93, we assumed that the ionization frequency \(v'\) is density independent. In many cases, however, this is not so. At low densities the dependence of \(v'\) on \(n\) may be due to the effect of electron-electron collisions. As shown in Section 5.6 these collisions tend to "maxwellize" the distribution, increasing the number of fast electrons, that is, the ionization frequency. At high concentrations stepwise ionization also plays an important part; its efficiency depends quadratically on the density. If the dependence of \(v'\) on \(n\) is allowed for, the balance equation 7.93 becomes nonlinear. The density distribution described by this equation depends on the form of the dependence of \(v'\) on \(n\). Figure 7.4 (curve 2) presents, by way of example, the distribution \(n(r)\) for \(v' \sim n\). The particle balance condition obtained by solving the equation can be written in a form similar to Eq. 7.96: \(v'(n) = D_A/\Lambda^2\), where \(v'(n_0)\) is the ionization frequency at the maximum electron density. The ratio between the diffusion length \(\Lambda\) and the plasma radius naturally differs from Eq. 7.95. At \(v'n\), for instance, \(\Lambda = a/3.5\).

The equality 7.96 makes it possible to find the electron temperature. It can be found if the dependence of \(v'\) on \(T_e\) is known. For the Maxwellian distribution this dependence is given by Eq. 5.117. Substituting it into Eq. 7.96, we obtain the transcendental equation for \(T_e\). At \(T_e \ll \bar{E}\) it has the form

$$2 \sqrt{\frac{2}{\pi}} \frac{T_e}{m_e} n_0 \delta \exp \left( -\frac{\bar{E}}{T_e} \right) = \frac{T_e}{m_e \nu_n \Lambda^2}$$  \hspace{1cm} (7.98)

where \(\delta\) is determined in conformity with Eq. 2.86. Its approximate solution yields

$$T_e = \left[ \ln \left( \frac{\Lambda^2}{\pi a \bar{E} m_e \nu_n} \frac{m_e \nu_n}{\bar{E}} \right) \right]^{-1}$$

(here \(\Lambda^2 = 1/n_0 \delta\)).

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where \(\delta\) is determined in conformity with Eq. 2.86. Its approximate solution yields

$$T_e = \left[ \ln \left( \frac{\Lambda^2}{\pi a \bar{E} m_e \nu_n} \frac{m_e \nu_n}{\bar{E}} \right) \right]^{-1}$$

(here \(\Lambda^2 = 1/n_0 \delta\)).

The solution of the particle balance equation 7.93 made it possible to find the radial density distribution, but did not yield its absolute value, that is, \(n_0\). In the case at hand the concentration depends on the longitudinal discharge current. The current density is related to the external field via the electron mobility (see Eq. 7.16) \(j = e \nu_0 E_0\). Accordingly, the total current can be calculated by integrating over the plasma cross section \(I = \int j dS = 2ae \nu_0 E_0 \int n(r)r dr\). Substituting Eq. 7.94, we find

$$n_0 = \frac{2.3 f}{\pi a^2 e \nu_0 E_0} = \frac{2.3 m_e \nu_n}{\pi a^2 \nu_0 E_0} \frac{1}{E_0}$$  \hspace{1cm} (7.99)

So far we have neglected the effect of volume recombination. When it is necessary to allow for the quadratic recombination the particle balance equation 7.92 takes the form

$$D_A \left( \frac{d}{dr} \frac{dn}{dr} \right) + v'n - \alpha n^2 = 0$$  \hspace{1cm} (7.100)

where \(\alpha = (s'clock)\) is the recombination coefficient. If the recombination effect is predominant, that is, \(\alpha n_0 \gg \nu_D \approx 1/\Lambda^2\), we can neglect the effect of the diffusion term of the equation in the central region of the plasma.

The resulting particle balance condition will be

$$v' = \alpha n_0$$  \hspace{1cm} (7.101)

It means that the charged particle production rate as a result of ionization is equal to the rate of their recombination removal. The right-hand side of (7.101) can be called the recombination removal frequency, and the reciprocal quantity, the recombination time

$$\tau_r = \frac{1}{v'} = \frac{1}{\alpha n_0}$$  \hspace{1cm} (7.102)
Since $\nu$ and $\alpha$ depend on $T_e$, the equality 7.101 defines the electron temperature relative to the density at the center. It is disturbed only near the plasma boundaries, where the diffusion removal of particles is substantial. It is easy to estimate the width of the region ($b$), where the diffusion is considerable, by comparing the first and last terms of (7.100). Assuming $\partial n / \partial r \sim n / b$ in this region, we obtain

$$\frac{D_a}{b^2} = \alpha n_0, \quad b = \frac{D_a}{\alpha n_0} \ll \Lambda.$$  

In this case the radial density distribution is plane in the central region and has steep gradients near the boundaries (see Fig. 7.4, curves 3, $\alpha n_0 = 2D_a/\Lambda^2$; and 4, $\alpha n_0 = 3D_a/\Lambda^2$). As the ratio of the diffusion coefficient to the recombination coefficient increases and so does $\nu / \nu_r$, the density gradients at the plasma boundaries flatten out, and at $\nu / \nu_r > \nu_r$ the density distribution approaches the diffusion distribution (Eq. 7.94).

Quantitatively, this transition can be traced using Eq. 7.100. We now pass to the electron energy balance. The balance equation 7.84 for the case in question ($E \perp \text{grad } n$; $\text{grad } T = 0$, $\partial T / \partial t = 0$) takes the form

$$-\text{div}(\kappa \text{grad } T_e) - D_a n T_e \text{ div} \left( \frac{\text{grad } n}{n} \right) = -\frac{3}{2} n \kappa_e \nu_a (T_e - T_a) + n e^2 E_0^2 \left( \frac{m_e}{m_e} \right),$$

(7.103)

With a given concentration distribution this equation is a nonlinear differential equation for the electron temperature. As noted above, the relative gradient of the electron temperature can usually be assumed much less than the relative density gradient. Assuming also that the conditions 7.85 are met, we neglect the terms on the left-hand side, as a first approximation. The energy balance equation then amounts to the equality of the average energy acquired by the electrons in the longitudinal electric field and the energy lost by them on elastic and inelastic collisions (Sections 5.4 and 5.6):

$$n e^2 E_0^2 = \frac{3}{2} n \kappa_e \nu_a (T_e - T_a)$$

whence we find the relationship between the strength of the external electric field and the electron temperature:

$$E_0^2 = \frac{3}{2} e^2 \kappa_e \nu_a^2 (T_e - T_a)$$

(7.104)

Note that this relationship is determined not only by the factor $T_e - T_a$; the average energy transfer coefficient $\kappa_e$ and the average collision frequency $\nu_a$ may also depend on the electron temperature.

A more accurate condition of the balance of average energies can be obtained from Eq. 7.103. To do this, we must integrate the equation of the plasma cross section. In all the terms, except the first one, we can, as before, assume the electron temperature to be radius independent. We integrate, assuming the radial density distribution to be a diffusion distribution (Eq. 7.94). The integral of the first term is found from the boundary condition (Eq. 7.88):

$$J_1 = \int \text{div } q_e \, ds = 2\pi a \rho_a = 2\pi a T_e \left( 2 + \ln \left( \frac{m_e}{m_i} \right) \right)$$

and, further, since the flux of the particles at the boundary $\Gamma_s$ is determined by their diffusion $2\pi a \Gamma_s = (D_a/\Lambda^2) \int n \, ds$, we obtain

$$J_1 = \frac{D_a}{\Lambda^2} n T_e \pi a^2 \left( 2 + \ln \left( \frac{m_i}{m_e} \right) \right)$$

($\bar{n} = 1/\pi a^2 \int n \, ds$ is the density averaged over the cross section).

The integral of the second term transforms to

$$J_2 = D_a T_e \int \text{div} \left( \frac{\text{grad } n}{n} \right) \, ds = D_a T_e \int \left[ \Delta n - \frac{(\text{grad } n)^2}{n} \right] \, ds$$

(7.105)

The integrals of the third and fourth terms include the quantity $\int n \, ds = \pi a^2 \langle n \rangle$.

Collecting the four terms, we obtain the average-energy balance condition:

$$\frac{e^2 E_0^2}{m_e \nu_a} \approx \kappa_e \nu_a (T_e - T_a) + D_a \frac{1}{\Lambda^2} n T_e \left( 2 + \ln \sqrt{\frac{m_i}{m_e}} + \ln \frac{n_0}{n_s} \right)$$

(7.105)

This equality differs from Eq. 7.104 by the second term, which takes into account the energy losses due to the removal of particles to the walls. Since the factor $D_a/\Lambda^2 = \nu_r$ determines the reciprocal diffusion time, the
average energy losses per one removed electron are equal to

$$\xi = 3T_e + T_e \ln \left( \frac{n_0}{n_e} \right) + T_e \ln \sqrt{m_e/m_i}$$  \hspace{1cm} (7.106)$$

This sum includes, firstly, the energy carried away by the electrons directly to the walls. As shown in Section 7.5, the average value of this energy with a Maxwellian distribution is $2T_e$. Secondly, it includes the energy spent on maintaining the ambipolar field in the plasma. This energy is approximately equal to $\varepsilon (\varepsilon_e - \varepsilon_0) = T_v \ln (n_0/n_e)$, where $\varepsilon_e - \varepsilon_0$ is the ambipolar potential difference (Eq. 7.66); it accelerates the ions in their motion toward the boundaries. Finally, the sum 7.106 includes the energy expended on maintaining the wall-layer potential difference $\Delta \varepsilon = (T_e/e) \ln \sqrt{m_i/m_e}$ (Eq. 7.81). As noted above (see p. 207), this energy goes to accelerate the ions, which carry it to the wall.

Let us now obtain the ion temperature, assuming that the condition 7.90 is met and the energy transfer by the ions is insignificant. Making use of the energy balance equation 7.91 for a stationary plasma, we get

$$T_i - T_v = \frac{m_i}{3} b_i^2 E_o^2 + \frac{1}{3 m_i} \left[ \frac{1}{n} \frac{\partial (D_i n)}{\partial r} \right]^2$$

and with a velocity-independent ion collision frequency,

$$T_i - T_v = \frac{4}{3} \frac{e^2 E_o^2}{m_i \nu_{ia}} + \frac{T_i^2}{3} \frac{1}{m_i \nu_{ia}} \left( \frac{1}{n} \frac{\partial n}{\partial r} \right)^2$$

$$= \frac{4}{3} \left( \frac{e^2}{m_i \nu_{ia}} \right) (E_o^2 + E_i^3)$$  \hspace{1cm} (7.107)$$

where $E_o = (T_v/n) \, dn/dr$ is the ambipolar field (Eq. 7.66).

The ion temperature is determined by the balance between the energy received by the ions in the electric field and the energy transferred by them to the neutral atoms on elastic collisions. Expression 7.107 shows that the ion heating is accomplished both by the external field $E_o$ and the ambipolar field $E_i$. It is easy to estimate, with the aid of equality 7.107, the relationship between these fields:

$$\frac{E_i}{E_o} = \frac{T_i}{T_v} \frac{1}{n} \frac{\partial n}{\partial r} \frac{\sqrt{\kappa_e}}{\sqrt{\kappa_ia}} \frac{1}{n} \frac{\partial n}{\partial r} = \frac{1}{\sqrt{\kappa_ia}} \frac{\lambda_{ia}}{\lambda_e} \frac{n_0}{n}$$

This estimate shows that at a sufficiently high value of $\lambda_{ia}$ the ambipolar field can be compared with the longitudinal one. Then the ion temperature must rise from the central region to the plasma boundaries, since the ambipolar field $E_i \sim 1/n$ increases in that direction.

It is easy to ascertain that ion heating, which is determined by the expression 7.107, is much weaker than that of the electrons. Comparing Eqs. 7.107 and 7.104, we obtain

$$\frac{T_i - T_v}{T_i - T_v} = \frac{\kappa_{ia} m_i \nu_{ia}^2}{m_e \nu_{ea}^2} = \frac{\lambda_{ia}^2}{\lambda_e} \frac{h_e}{h_{ia}}$$

The difference in heating is due primarily to the difference in energy transfer in collisions: the fraction of energy transferred by the electron ($\kappa_{ia} \ll 1$) is always much less than the ion energy losses ($\kappa_{ia} = 1$).

Thus the particle and energy balance equations make it possible to establish the main characteristics of a gas-discharge plasma. When direct ionization of the atoms by the electrons is the basic ionization process, charged particles are mainly removed through ambipolar diffusion, and the transfer only slightly affects the particle energy balance, we obtain expressions for these characteristics. Thus the density distribution $n(r)$ (Eq. 7.94) is found by solving the particle balance equation, the concentration $n_0$ is determined by the discharge current (Eq. 7.99), and the electron temperature $T_e$ by the particle balance condition (Eq. 7.96); the strength of the field in the discharge $E_o$ is obtained from the electron energy balance (Eq. 7.104), and the ion temperature $T_i$ from the ion energy balance (Eq. 7.107).

7.7 Ionization Instability

The balance of the charged particles in a plasma of a stationary gas discharge considered in Section 7.6 may turn out to be unstable. The instability of the ionization balance is caused by the dependence of the ionization frequency on the electron density. As noted above, this dependence may be due to the effect of electron-electron collisions on the distribution function and to stepwise ionization. If the effect of these processes is substantial, an accidental increase in electron density in some section of the plasma column may lead to a local increase of ionization frequency and to a further growth of density. As a result it will increase exponentially with time. This increase of a small perturbation points to instability of the ionization equilibrium with respect to the perturbation.

Let us consider the conditions for ionization instability. To this end we use the particle balance equation (see Eq. 7.77):

$$\frac{dn}{dt} - D_i \Delta n = n' n$$  \hspace{1cm} (7.108)$$

Assume that a plasma exhibits a small density perturbation depending harmonically on the longitudinal coordinate. With this perturbation the
density distribution can be represented as

\[ n(r, z, t) = n^0(r) + n^{(1)}(r, z, t) \]

where \( n^0(r) \) is the unperturbed density, \( n^{(1)}(r, x, t) \) is the density perturbation expressed in a complex form to simplify the analysis, and \( k \) determines the longitudinal scale or the wavelength of the perturbation \( \lambda_k = 2\pi/k \). The density perturbation changes the electric field strength distribution and causes nonuniformity in electron heating. Accordingly, the electron temperature is also perturbed. Since, however, the electron thermal conductivity of the plasma is high, this perturbation is small in a definite range of conditions. We neglect it for the time being.

Substituting the density of Eq. 7.109 into Eq. 7.108 and assuming the radial distributions of the stationary density and of the perturbation to be equal, we obtain the following equation for the density perturbation:

\[ \frac{\partial n^{(1)}}{\partial t} + k^2 D_{\alpha} n^{(1)} = n^0 (\nu'_{\alpha} - \nu') \]  

(7.110)

where \( \nu'_{\alpha} \) is the average ionization frequency in the presence of a perturbation, and \( \nu' \) in its absence. Assuming the perturbation to be small and restricting ourselves to the first term of the expansion \( \nu'_{\alpha} \) in its powers, we get

\[ \nu'_{\alpha} - \nu' = \frac{\partial \nu'_{\alpha}}{\partial n} n^{(1)} \]

The perturbation equation then takes the form

\[ \frac{\partial n^{(1)}}{\partial t} = (\frac{n \partial \nu'}{\partial n} - k^2 D_{\alpha}) n^{(1)} \]

(7.111)

Its solution, with a positive coefficient on the right-hand side, yields exponentially increasing perturbation

\[ n^{(1)} = n_0^{(1)} \exp(\gamma t); \quad \gamma = n \frac{\partial \nu'}{\partial n} - k^2 D_{\alpha} \]

(7.112)

The quantity \( \gamma \) is called the instability increment. Thus the criterion of instability in the model under consideration amounts to the inequality

\[ \frac{n \partial \nu'}{\partial n} > k^2 D_{\alpha} \]  

(7.113)

With the opposite inequality the power exponent becomes negative. This means that an accidental perturbation decreases with time and hence the plasma is stable with respect to the perturbation.

It is easy to understand the physical meaning of Eq. 7.113. Its left-hand side describes the increase in ionization frequency which perturbs the ionization equilibrium in the presence of a perturbation, and the right-hand side determines the rate of diffusion spread of the perturbation \( \nu_0 \sim D_{\alpha}/(\lambda_k^2 - D_{\alpha} k^2) \). Therefore, the inequality 7.113 means that the rate of ionization increase in perturbation exceeds that of its diffusion spread. The criterion 7.113 limits the perturbation from below. Taking into account the stationary relationship between the ambipolar diffusion coefficient and the ionization frequency (Eq. 7.96), we reduce this criterion to

\[ k^2 \lambda^2 < n \frac{\partial \nu'}{\partial n} \]

(7.114)

Since the right-hand side of Eq. 7.114 is of the order of unity, it indicates that instability can arise only at perturbation wavelengths \( \lambda_k = 2\pi/k \) exceeding the plasma column radius.

Let us now consider the effect of a perturbation in electron temperature on the conditions of development of the instability. This effect may be considerable even for relatively small perturbations \( T^{(1)}/T \ll n^{(1)}/n_0 \), because the dependence of the ionization frequency on the electron temperature is usually much stronger than its dependence on the density. The perturbation of the electron temperature is due to the longitudinal electric field, which determines the electron heating. In accordance with Eq. 7.63 the longitudinal field consists of two components: the current component \( E_j \) and the ambipolar one \( E_A \). The current field is given by the condition of constancy of the longitudinal current along the length

\[ j = e n b \nu \]

(7.115)

Assuming for simplicity that the collision frequency of the electrons and, hence, their mobility \( b = e/m_e \nu_\alpha \) are independent of temperature, we obtain

\[ n E_j = (n^{(0)} + n^{(1)}) (E_0 + E_j^{(1)}) \]

where the field \( E_j \) is represented as a sum of the unperturbed component \( E_0 \) and the perturbation \( E_j^{(1)} \). Assuming the perturbations to be small, \( n^{(1)} \ll n^{(0)} \) and \( E_j^{(1)} \ll E_0 \), we find the relationship of the perturbations of the field \( E_j \) and the density \( n \):

\[ E_j^{(1)} = \left( \frac{n^{(1)}}{n^{(0)}} \right) E_0 \]

(7.116)

which characterizes the distribution of \( E_j^{(1)} \) along the length.
As can be seen, the perturbation of $E_1^{(0)}$ is in counterphase with the density perturbation: an increase in density reduces the field (Fig. 7.5). The ambipolar component of the longitudinal electric field is given by Eq. 7.66; $E_A = -(T_e / e n) \partial n / \partial z$. Substituting Eq. 7.109, we have

$$E_A^{(0)} = i k \frac{T_e}{e n} n^{(0)}$$  (7.117)

The longitudinal distribution $E_2^{(0)}(z)$ is shifted in phase by a quarter wavelength with respect to the density distribution $n^{(0)}(z)$—the field is at a maximum in the region where the density gradient is at a maximum (see Fig. 7.5). The relations obtained determine the power of heating of the electron gas with an allowance for the field perturbations. Accordingly, the power released in a unit volume,

$$P_B = j_b E_b = en_b E_0 (E_0 + E_1^{(0)} + E_2^{(0)})$$
$$= en_b E_0^2 - en^{(0)} b E_0 + ik b T_e E_0 n^{(0)}$$  (7.118)

where the constancy of the longitudinal current (Eq. 7.115) is taken into account. The first term of this equation describes longitudinally uniform heating in the absence of perturbations and the second and third, the change in heating due to the density perturbation. The resulting perturbations in electron temperature can be found from the energy balance equation 7.84, which we write approximately in the form

$$\frac{3}{2} n \frac{\partial T_e}{\partial t} - \text{div}(\mathcal{K} \text{ grad } T_e) = P_e - \frac{3}{2} \kappa_e n \nu a n T_e$$  (7.119)

where at $\nu = \text{const}$, $\mathcal{K} = \frac{1}{2}(T_n / m \nu a_n)$. Without perturbations the left-hand side of the equation vanishes, and the energy balance condition amounts to the equality

$$P_B = en_b E_0^2 = \frac{3}{2} \kappa_e n \nu a n T_e$$

It yields the relationship of the strength of the longitudinal field with the temperature (Eq. 7.104):

$$eE_0 = \left( \frac{3}{2} \kappa_e n \nu a \frac{T_e}{T_e} \right)^{1/2} = \sqrt{\frac{3}{2} \kappa_e T_e}$$  (7.120)

where

$$\lambda_{T_e} = \left( \frac{T_e}{\kappa_e n \nu a} \right)^{1/2} = \lambda_e \sqrt{\kappa_e}$$

is the thermal relaxation length, that is, the length over which energy is exchanged between the electrons and atoms.

The temperature perturbation must be sought in a form similar to Eq. 7.109:

$$T^{(0)}(z, t) = \text{Re}(T^{(0)}(t) \exp(-ikz))$$

When substituting it into Eq. 7.119, we take into account that owing to the high thermal conductivity of the electron gas the first term describing the change in perturbation $T^{(0)}$ with time is usually much less than the second term $n \partial T^{(0)} / \partial t \approx \mathcal{K} / k |T^{(0)}|$ and the relative temperature perturbation is much less than the relative density perturbation $|T^{(0)}/T| \ll |n^{(0)}/n|$. *Neglecting the corresponding small terms, we get

$$k^2 \mathcal{K} T_e T^{(0)} = P^{(0)} - \frac{3}{2} \kappa_e n \nu a T_e n^{(0)}$$  (7.121)

Using the relation 7.118 for $P_B$ and taking into account Eq. 7.120, we find

*As follows from Eq. 7.122, the smallness conditions $|T^{(0)}/T|$ amount to the inequality $k \gg eE_0 T_e = 1/\lambda_e$. 

---

**Fig. 7.5** Plasma parameters under ionization instability conditions.
the relationship of the temperature and density perturbation:

\[
\frac{T_1 - T_0}{T_0} = 4 \left( \frac{e^2 E_0}{5 k^2 T_0} - \frac{e E_0}{k T_0} \right) n^{(1)} \tag{7.122}
\]

\[
\frac{T_1 - T_0}{5 k^2 T_0} = -\frac{6}{5} \left( \frac{\nabla T}{5 k^2 T_0} \right) n^{(1)}
\]

which allows for the fact that at \( n_0 = \text{const} \), \( \nabla T = \frac{3(n T/e)}{T} \).

The perturbation of the electron temperature, as well as the density perturbation, changes the ionization frequency. Taking both perturbations into consideration, we obtain

\[
\nu' - \nu = \frac{\partial \nu}{\partial n} n^{(1)} + \frac{\partial \nu}{\partial T} T_1^{(1)}
\]

where \( T_1^{(1)} \) satisfies Eq. 7.122. Substituting this difference into the particle balance equation 7.110, we reduce it to the form

\[
\frac{\partial n^{(1)}}{\partial t} = \frac{1}{5} \frac{\nabla T}{k T_0} + \frac{\partial \nu}{\partial n} \frac{\partial T}{T} n^{(1)} + \frac{\partial \nu}{\partial T} T_1^{(1)} \tag{7.123}
\]

The solution of the equation yields the time dependence of \( n^{(1)} \)

\[
\nu^{(1)} = n_0 \exp(i \omega t) \exp(i k z)
\]

where

\[
\omega = \frac{\sqrt{6}}{5} \frac{T_0}{k T} \frac{\partial \nu}{\partial n}
\]

\[
\gamma = n \frac{\partial \nu}{\partial n} + D_\nu k^2 \frac{6}{5} \frac{T_0}{k T} \frac{\partial \nu}{\partial T}
\]

Accordingly we obtain

\[
n^{(1)} = \text{Re}[n^{(1)} \exp(-ikz)] = n_0 \exp(i \omega t) \cos(\omega t - k z) \tag{7.125}
\]

As seen from this equation, the density perturbation (as well as the associated field temperature perturbations) is a wave of frequency \( \omega \) propagating in the direction of the \( z \) axis. It is easy to understand its propagation mechanism. As has already been noted, the ambipolar electric field proportional to \( \nabla n \) is shifted in phase by a quarter wavelength relative to the density perturbation (see Fig. 7.5). The electron heating and the ionization frequency increase with the electric force \( eE \). Accordingly, the density increases in the region where \( eE > eE_0 \) and the ionization frequency exceeds that of diffusion \( \nu' > \nu_0 \) and decreases in the region where \( eE < eE_0 \) and \( \nu' < \nu_0 \). Since the regions of the maximum and minimum fields are shifted at each given instant relative to the density maxima and minima, the change of density in them results in a displacement of the maxima and minima, that is, in ionization wave propagation (see Fig. 7.5).

The equation for the instability increment \( \gamma \) (Eq. 7.124), with due regard for the temperature perturbation, differs from Eq. 7.112 by an additional term, which reduces the increment. It is associated with the perturbation of the current electric field \( E_j \) (Eq. 7.116). Since this perturbation is in counterphase with the density perturbation, it reduces the electron temperature and the ionization frequency in the region of the density maxima and increases them in the region of the density minima. Such changes reduce the initial perturbation. This effect is the greater, the larger the perturbation wavelength, because with decreasing wavelength the electron thermal conductivity decreases the temperature perturbation (as can be seen from Eq. 7.124 the last term is proportional to \( 1/k^2 \)).

An analysis of the expression for \( \gamma \) enables us to determine the conditions for ionization instability. Since the diffusion term in \( \gamma \) is proportional to \( k^2 \) and the temperature term is inversely proportional to it, the maximum of \( \gamma \) is obtained at a \( k \) such that these terms are equal:

\[
k_0^2 = \frac{6}{5} \frac{T_0}{D_\nu} \frac{\partial \nu}{\partial T}
\]

\[
\gamma = \frac{1}{2} \frac{\lambda_T}{\nu} \left( \frac{n \partial \nu}{\partial T} \frac{T_0}{\nu} \right)^{\frac{1}{2}} \tag{7.126}
\]

Here \( \gamma \) is positive if \( n \partial \nu / \partial n > 2D_\nu k_0^2 \) or

\[
\lambda < \frac{1}{2} \frac{T_0}{\nu} \left( \frac{n \partial \nu}{\partial T} \frac{T_0}{\nu} \right)^{-1/2}
\]

The inequality 7.127 shows that instability is possible only with a sufficiently small plasma radius:

\[
a = \frac{\text{const}}{n_0 e^2 \nu_0^2 \lambda_T}
\]

If Eq. 7.127 is fulfilled with a large safety margin, there is a wide range of \( k \) values within which ionization instability can arise. On the side of large \( k \) this region is bounded by Eq. 7.114, which takes into account the effect of diffusion spread of the perturbation. On the side of small \( k \) it is bounded by the effect of temperature perturbation. In accordance with Eq. 7.124

\[
n \frac{\partial \nu}{\partial n} > \frac{T_0}{k^2 \lambda_T \nu T}
\]

or

\[
k^2 \lambda_T > \left( \frac{T_0}{\nu T} \frac{\partial \nu}{\partial T} \right) \left( \frac{n \partial \nu}{\partial T} \right)^{\frac{1}{2}}
\]
The maximum increment is determined by the first term in the expression for \( \gamma \) (see Eq. 7.124):

\[
\gamma_{\text{max}} = \frac{n \partial \nu}{\partial n} \tag{7.129}
\]

The foregoing analysis makes it possible to determine the conditions for the stability of the ionization balance of a gas-discharge plasma with respect to small perturbations. It is based on the linearization of the balance equations, which is valid while the perturbations of the plasma parameters are much less than their stationary values. In the parameter region where the ionization equilibrium is unstable an increase in perturbation may lead to a new stationary state characterized by a substantial change in plasma parameters. When considering such changes no simplifications associated with the linearization of the equations are allowed, and a much more complicated nonlinear analysis is necessary. Without dwelling on it, we just note that ionization instability may result in a strong longitudinal modulation of the main parameters of a gas-discharge plasma. As a consequence, standing or traveling nonlinear waves are formed in the plasma column, which are called striations.

7.8 PLASMA DECAY

We now consider, with the aid of the balance equations, the decay (deionization) of a plasma set up in a long cylindrical chamber under the effect of an external electric field or some other ionization sources. After these are cut off, the temperature of the charged particles decreases, since the energy losses are not offset by external sources. With decreasing temperature the ionization efficiency drops sharply, and diffusion and recombination reduce the concentration of the charged particles (Fig. 7.6). The plasma at this stage is described as decaying.

To obtain the decay characteristics we take advantage of the particle and energy balance equations. As in the preceding section we assume that the main mechanism of charged particle removal is the ambipolar diffusion and that energy transfer by electrons and ions is insignificant (i.e., the inequalities 7.85 and 7.90 are valid). The balance equations 7.77, 7.86, and 7.91 then take the form

\[
\frac{\partial n}{\partial t} - D_n \nabla n = 0;
\]

\[
\frac{\partial T_e}{\partial t} = -\kappa_{ea}\nu_{ea}(T_e - T_a);
\]

\[
\frac{\partial T_i}{\partial t} = -\frac{1}{2} \nu_a(T_i - T_a) \tag{7.130}
\]

In the first one we omitted the term \( v'n' \), assuming that \( v' \ll D_n/A^2 \), which means that during decay ionization practically does not affect the particle balance. At velocity-independent collision frequencies \( \nu_{ea} \) and \( \nu_a \) with a coefficient \( \kappa_{ea} \) the solutions of the energy balance equations can be written thus:

\[
T_e = T_{ea} + (T_{eo} - T_{ea}) \exp(-\kappa_{ea}\nu_{ea}t);
\]

\[
T_i = T_{ia} + (T_{io} - T_{ia}) \exp\left(-\frac{1}{2} \nu_{ia} t\right) \tag{7.131}
\]

where \( T_{ea} \) and \( T_{ia} \) are the electron and ion temperatures at \( t = 0 \).

It is seen from Eq. 7.131 that after the external energy sources are cut off the electron and ion temperatures vary exponentially. As a result of collisions with neutral atoms their temperatures in the course of decay approach (relax to) the temperature of the neutral gas. The time constants of the temperature drop are equal to

\[
\tau_{te} = \frac{1}{\kappa_{ea}\nu_{ea}}; \quad \tau_{ti} = \frac{2}{\nu_{ia}} \tag{7.132}
\]

The relaxation time of the electron temperature greatly exceeds that of the ion temperature (\( \tau_{te} \gg \tau_{ti} \)). This difference is due to the low efficiency of electron energy transfer on collisions with atoms (\( \kappa_{ea} \ll 1 \), whereas \( \kappa_{ia} = 1 \).

When \( \nu_{ea} \), \( \nu_{ia} \), and \( \kappa_{ea} \) are temperature dependent the relaxation law naturally differs from the exponential one. For electrons, say, at \( \kappa_{ea} \rightarrow T_e^2 \) and \( T_e \gg T_{eo} \), the energy balance equation takes the form

\[
\frac{\partial T_e}{\partial t} = -(\kappa_{ea}\nu_{ea})\left(\frac{T_{eo}}{T_e}\right)^2 T_e \]
Its solution leads to the following \( T_c(t) \) dependence:
\[
T_c = T_0 [1 + s(\kappa_{\alpha} \nu_{\alpha}) t]^{1/r}
\]
(7.133)
where \((\kappa_{\alpha} \nu_{\alpha})\) is the value of \( \kappa_{\alpha} \nu_{\alpha} \) at \( t = 0 \).

We now consider the solution of the particle balance equation. For a cylindrically symmetric plasma it can be written as
\[
\frac{\partial n(r, t)}{\partial t} - D_A(t) \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n(r, t)}{\partial r} \right) = 0
\]
(7.134)
where the diffusion coefficient \( D_A = 2(T_e + T_i)/m_\nu_{\alpha} \) is, generally speaking, a function of time.

This linear partial differential equation is sometimes called the diffusion equation. Its solution is well known. A partial solution of the equation has the form
\[
g(r, t) = \exp \left[ -\frac{\xi^2}{a} \int_0^t D_A(t') dt' \right] J_0 \left( \frac{\xi}{a} r \right)
\]
This solution meets the boundary conditions, that is, it vanishes at \( r = a \) if \( \xi \) is a root of the Bessel function \( J_0(\xi) = 0 \). Since the number of such roots is infinite we obtain an infinite number of independent solutions. The general solution can be represented as their linear combination:
\[
n(r, t) = \sum C_k \exp \left[ -\frac{\xi_k^2}{a} \int_0^t D_A(t') dt' \right] J_0 \left( \frac{\xi_k}{a} r \right)
\]
(7.135)
The constant coefficients \( C_k \) are determined by the initial density distribution:
\[
C_k = \int_0^a n(r, 0) J_0 \left( \frac{\xi_k}{a} r \right) r dr
\]
In particular, if the initial distribution is a diffusion distribution (see Eq. 7.94), \( n = n_0 J_0(r/A) \), where \( A = a \xi_1 \) and \( \xi_1 = 2.405 \) is the first root of the Bessel function, the sum 7.135 retains only one term.

Thus the diffusion distribution established in the stationary discharge remains unchanged on plasma decay. The arbitrary initial distribution is deformed in the course of decay. Indeed, the power exponents in Eq. 7.135 depend on the number \( k \), since \( \xi_k \) increases with \( k \). Therefore, after a sufficiently long time \( t > \tau_0 = A^2/D_A \), the first term with the lowest index greatly exceeds the others, and the solution can be written approximately as follows:
\[
n(r, t) = C_1 \exp \left[ -\frac{1}{A^2} \int_0^t D_A dt' \right] J_0 \left( \frac{r}{A} \right)
\]
(7.136)
This equation shows that irrespective of the initial density distribution, after a time of the order of \( \tau_0 \) a diffusion concentration distribution described by the zero-order Bessel function is established.

In accordance with the equations obtained the decrease in charged particle density in the course of plasma decay is determined by the ambipolar diffusion coefficient \( D_A = 2(T_e + T_i)/m_\nu_{\alpha} \). Its time dependence is associated with the time variation in electron and ion temperatures. For example, when \( \nu_{\alpha}, \nu_{\beta}, \text{and} \kappa_{\alpha} \) are velocity independent, one can use Eq. 7.131 to find the dependence \( D_A(t) \). The charged particle diffusion time \( \tau_0 \) usually greatly exceeds the relaxation time of the electron temperature \( \tau_T \):
\[
\frac{\tau_0}{\tau_T} = \frac{\Lambda^2}{D_A} \kappa_{\alpha} \nu_{\alpha} = \sqrt{\frac{m_e}{m_i} \kappa_{\alpha} \Lambda^2} > 1
\]

The time \( \tau_T \), in turn, exceeds the relaxation time of the ion temperature (Eq. 7.132). Therefore in the first stage of decay the electron and ion temperatures decrease, while the density remains practically the same. In the next stage (at \( t > \tau_T \)) the temperature of the charged particles is virtually equal to that of the neutral gas, and the density decreases relatively slower. In this stage the ambipolar diffusion coefficient is practically constant \( D_A = 4\pi \Lambda/m_\nu_{\alpha} \). Substituting it into Eq. 7.135 and 7.136, we obtain the law of density variation. In a sufficiently late stage of decay \( t > \tau_{D_i} \) when the diffusion distribution of densities is established, this law becomes exponential. From Eq. 7.136 we obtain
\[
n_0(t) = n_0(0) \exp \left( -\frac{D_A t}{\Lambda^2} \right)
\]
(7.137)
The time constant of plasma decay in this stage is equal to the diffusion time (Eq. 7.97):
\[
\tau_{D} = \frac{D_A}{\Lambda^2} = \frac{23.2 \ T_e}{a^2 \ m_\nu_{\alpha} \ k_{\alpha}}
\]
(7.138)
Under conditions where, along with the diffusion, the quadratic recombination substantially affects the plasma decay, the particle balance equation takes the form
\[
\frac{\partial n}{\partial t} - D_A \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n}{\partial r} \right) = -an^2
\]
(7.139)
and in the late stage of decay we can assume that the coefficients of diffusion \( D_A \) and recombination \( \alpha \) are constant and correspond to \( T_e = T_i = T_T \).

It can be shown that, with constant coefficients \( D_A \) and \( \alpha \), Eq. 7.139 has a partial solution in which the radial density distribution is, at each
given instant, close to the radial distribution in a stationary plasma, which is determined by the solution of Eq. 7.100. This distribution depends on the ratio of the diffusion and recombination removal frequencies \( \nu_D = D_\alpha / \Lambda^2 \) (Eq. 7.97) and \( \nu_r = \alpha n_0 \) (Eq. 7.102) and changes in the course of decay from a "flattened" distribution at \( \nu_r > \nu_D \) to a diffusion distribution at \( \nu_D \) (see Fig. 7.4). The change in maximum density \( n_0 \) with time, corresponding to this solution, is approximately described by the equation

\[
\frac{d n_0}{dt} = -\rho_D \frac{D_\alpha}{\Lambda^2} n_0 - \rho_r \alpha n_0 \]

where the coefficients \( \rho_D \) and \( \rho_r \) (of the order of unity) depend on \( \nu_m \nu_r \). The solution of this equation has the form

\[
n_0(t) = \frac{n_0(0)}{1 + \alpha n_0(0)t} \quad (7.140)
\]

At \( D_\alpha / \Lambda^2 \gg \alpha n_0 \) the solution 7.140 changes into an exponential dependence characteristic of diffusion decay. At \( \alpha n_0 \gg D_\alpha / \Lambda^2 \) the particle balance equation can be written as

\[
\frac{\partial n}{\partial t} = -\alpha n^2 \quad (7.141)
\]

or \( 1/n(t) = 1/n(0) + \alpha t \).

Thus the change in charged particle density in the late stage of plasma decay is characterized by ambipolar diffusion and recombination coefficients at a temperature of charged particles close to that of the neutral gas. Therefore the plasma decay is often used for measuring these coefficients.

7.9 DIRECTED MOTION IN HIGHLY IONIZED PLASMAS

In a highly ionized plasma one must take into account not only collisions of charged particles with neutrals, but also their collisions with each other. Then the quasi-stationary equations of electron and ion motion can be written as follows (see Eq. 6.62):

\[
-eE - \frac{1}{n} \text{grad}(nT_e) + R_{ee} + R_{ea} + R_{ad} + R_{d} + R_{a} = 0
\]

\[
+eE - \frac{1}{n} \text{grad}(nT_i) + R_{ea} + R_{ia} + R_{ad} + R_{d} = 0 \quad (7.142)
\]

They hold for the conditions 7.5 when the changes in directed velocities during the time between collisions can be neglected. In Eq. 7.142 \( R_{e} \) is the friction force acting on \( \alpha \)-type particles as a result of their collisions with \( \beta \)-type particles; in reference systems where the directed velocity of the neutral atoms is zero,

\[
R_{ea} = -m_e \nu_a u_e; \quad R_{ad} = -m_d \nu_a u_d; \quad R_{d} = -R_{e}\]

(7.143)

The quantities \( R_{ea} = g^e_{\alpha \beta} T_e \), where \( g^e_{\alpha \beta} = (T_{\alpha e}, T_{\alpha d}) \) are the components of the thermal force acting on the \( \alpha \)-type particles due to their collisions with \( \beta \)-type particles. As follows from the momentum conservation law on collisions, \( R_{e} = -R_{e} \).

Let us find the numerical values of the effective collision frequencies and the coefficients \( g^e_{\alpha \beta} \) for the case where the frequencies of collisions of electrons and ions with neutral atoms are velocity independent and these collisions make no contribution to the thermal force \( (\nu_{ea} = \text{const}; \nu_{ed} = \text{const}; g^e_{\alpha n} = g^e_{n \alpha} = 0) \). The frequency of electron–ion collisions (see Eq. 6.69),

\[
\nu_{e} = \frac{4 \pi n_{e}^{\alpha} u_{e}^{\alpha} L_{e}}{m_{e}^{\alpha} v_{e}^{\alpha}}
\]

can be assumed inversely proportional to the cube of the electron velocity, neglecting the contribution of the ions to the relative velocity and the weak velocity dependence of the coulomb logarithm. For this case the collision term representing the effect of electron–ion collisions on the directed velocity of the electrons when their distribution is near-Maxwellian can be found using Eqs. 6.48, 6.49, and 6.57 (in the eight-moments approximation):

\[
\left( \frac{\delta u_e}{\delta t} \right)_e = -\frac{m_e}{3 T_e} \langle \nu_{e} u \rangle (u_e - u) + \frac{1}{3 n T_e} \left( \nu_{e} \left( m_e u_e \right)^{\alpha} \left( m_e u_e \right)^{\alpha} \right) \quad (7.144)
\]

where \( \langle \cdot \rangle \) means averaging over the Maxwellian electron distribution. Carrying out this averaging, we obtain (see Eq. 5.116)

\[
\nu_{e} = \frac{m_e}{3 T_e} \langle \nu_{e} u \rangle
\]

\[
= 4 \pi \frac{n_{e}^{\alpha}}{m_{e}^{\alpha} v_{e}^{\alpha}} L_{e} \left( \frac{2}{\sqrt{\pi}} \right) \int_{0}^{\infty} v \exp\left( -\frac{m_{e} v^{2}}{2 T_e} \right) dv
\]

\[
= \frac{4 \sqrt{2 \pi}}{3} \frac{n_{e}^{\alpha}}{m_{e}^{\alpha} T_e} L_{e} \quad (7.145)
\]


\[
\left( v_\ne \left( \frac{m_{e}v^2 - m_{i}v_{ei}^2}{5T_{e}} \right) \right) = \frac{4\pi ne^4}{m_{e}} \sqrt{\frac{2}{\pi}} \left( \frac{m_{e}}{T_{e}} \right)^{5/2} \times \int_{0}^{\infty} dv \left( v - \frac{m_{e}}{5T_{e}} \right) \exp \left( -\frac{m_{e}v^2}{2T_{e}} \right) = \frac{9}{5} \tilde{v}_{ei}
\]

Equation 7.144 includes the electron heat flux. Substituting its value obtainable from Eq. 7.160, which takes into account the effects of electron-ion, electron-electron, and electron-atom collisions, we get

\[
m_{e} \left( \frac{8m_{e}}{9T_{e}} \right) = -m_{e} \tilde{v}_{ei} \nu_{ei} + 0.97 \tilde{v}_{ei} (u_{e} - u_{i}) - \frac{3}{2} \frac{\tilde{v}_{ei}}{v_{ei} + 1.87 \tilde{v}_{ei}} \text{ grad } T_{e}
\]

The first term of this equation yields the friction force,

\[
R_{d} = -m_{e} \nu_{ei} (u_{e} - u_{i}); \quad \nu_{ei} = \tilde{v}_{ei} \nu_{ei} + 0.97 \tilde{v}_{ei}
\]

and the second term the thermal force,

\[
R_{d}^t = -g_{T} \text{ grad } T_{e}; \quad g_{T} = \frac{3}{2} \frac{\tilde{v}_{ei}}{v_{ei} + 1.87 \tilde{v}_{ei}}
\]

It can be seen that the effective frequency of electron-ion collisions, which determines the friction force, and the coefficient \( g_{T} \), which determines the thermal force, depend not only on \( \tilde{v}_{ei} \), but also on \( \nu_{ei} \). In particular, if \( \nu_{ei} \gg \nu_{en} \), then \( \nu_{ei} = \tilde{v}_{ei} \) and \( g_{T} \approx \frac{3}{2} \tilde{v}_{ei} / \nu_{ei} \ll 1 \); at \( \nu_{ei} \gg \nu_{en} \) the equations of motion 7.142 take the form

\[
-eE = \frac{1}{n} \text{ grad } (nT_{e}) - g_{T} \text{ grad } T_{e} - m_{e} \nu_{ei} u_{e} - m_{i} \nu_{ei} (u_{e} - u_{i}) = 0
\]

\[
+eE = \frac{1}{n} \text{ grad } (nT_{e}) + g_{T} \text{ grad } T_{e} - \mu_{a} \nu_{ei} u_{e} - m_{i} \nu_{ei} (u_{e} - u_{i}) = 0
\]

where the collision frequency \( \nu_{ei} \) and the coefficient \( g_{T} \) are determined by Eqs. 7.146-7.148. It is taken into consideration here that in the electron and ion equations the collision terms due to electron-ion collisions are equal in magnitude and opposite in sign. Equations 7.149 enable us to find the directed velocity of the charged particles. As in a weakly ionized plasma, it can usually be represented as the sum of the ambipolar velocity and the velocity determined by the external electric field.

Let us first find the directed velocity due to the field. To do this, we put \( \nu_{ei} = g_{T} \text{ grad } T_{e} \) and \( \nu_{ei} \) in our equations. Then, adding them up, we obtain the ratio between \( u_{eff} \) and \( u_{eff} \):

\[
u_{eff} = \frac{m_{e} \nu_{eff}}{\mu_{a} \nu_{eff}} u_{eff} \]

which indicates that \( u_{eff} = u_{eff} \). Bearing this in mind, we find the directed electron velocity

\[
u_{eff} = \frac{-eE}{m_{e} (\nu_{ei} + \nu_{ei})}
\]

with the aid of Eq. 7.151 we write the current density and the plasma conductivity

\[
j = -neu_{eff} = \sigma E; \quad \sigma = \frac{ne^2}{m_{i} (\nu_{ei} + \nu_{ei})}
\]

The expression 7.152 is similar to Eq. 7.16 for the conductivity of a
weakly ionized plasma, the only difference being that it contains the summary collision frequency $\nu_{\text{eff}} + \nu_{\text{el}}$. For a highly ionized plasma, in which $\nu_{\text{el}} \gg \nu_{\text{eff}}$, we get, taking into account Eqs. 7.145 and 7.148,

$$\sigma = \frac{2ne^2}{m_e \tilde{\nu}_e} = \frac{3}{2\sqrt{2\pi}} \frac{T_e^{3/2}}{e^3 \nu_{\text{el}}^{1/2}}$$

(7.153)

The conductivity of such a plasma is practically independent of the charged particle density. It is easy to understand the cause. On the one hand, the conductivity is proportional to the number of current carriers, that is, to the electron density. On the other, it is inversely proportional to the density of ions, collisions which hinder electron acceleration. Since the electron and ion densities are equal, these dependences offset each other. The conductivity of a strongly ionized plasma depends only on the electron temperature, permitting the use of conductivity measurements for temperature determination.

Let us now find the ambipolar component of the directed velocity, assuming $u_e = u_i = u_A$ in Eq. 7.149. The friction force due to the electron–ion collisions reduces to zero, and the equations take a form similar to the equations of motion in a weakly ionized plasma (see Eq. 7.6). Adding up the equations of motion of the electrons and ions, we get the expression for the ambipolar velocity, which is similar to Eq. 7.74:

$$u_A = \frac{\operatorname{grad}[n(T_e + T_i)]}{\mu_{\text{ion}} n} = \frac{1}{n} \operatorname{grad}(D_A n)$$

(7.154)

(this takes into account that $m_e \nu_{\text{el}} \ll \mu_{\text{ion}} \nu_{\text{el}}$). The value of the ambipolar electric field will be found from the difference of the equations of motion (Eqs. 7.149):

$$E_A = \frac{-\operatorname{grad}(nT_e)}{en} \frac{\text{g}_T}{e} \operatorname{grad} T_e$$

$$= \frac{-T_e}{e} \frac{\operatorname{grad} n}{n} \left(1 + \frac{\text{g}_T}{e} \right) \operatorname{grad} T_e$$

(7.155)

Since Eqs. 7.151 and 7.154, which define the directed velocity of charged particles in a highly ionized plasma, are similar to the corresponding equations for a weakly ionized plasma, the particle balance equations are also similar, and we do not discuss them here.

The obtained expressions for the directed velocity fail when the mean free paths of the charged particles relative to their collisions with neutral atoms are comparable with or greater than the characteristic size of the plasma. Under such conditions the motion of the charged particles under the effect of the density and temperature gradients no longer bears a diffusion character; their directed velocity in the presence of gradients can be compared with the thermal velocity. Switching to the limits $\nu_{\text{el}} \to 0$, $\nu_{\text{eff}} \to 0$ in Eq. 7.149 and adding them up, we find that the total-pressure gradient must be equal to zero:

$$\operatorname{grad}[n(T_e + T_i)] = 0; \quad p = n(T_e + T_i) = \text{constant}$$

The constancy of the total plasma pressure is the condition for the applicability of Eqs. 7.149 in describing the directed motion in a fully ionized plasma. In the absence of gradients of the ion and electron pressure [$\operatorname{grad}(nT_e) = \operatorname{grad}(nT_i) = 0$] the two equations amount to the equality:

$$eE + 0.71 \frac{\text{g}_T}{e} \frac{\operatorname{grad} T_e}{T_e} = -0.51 \frac{m_e \tilde{\nu}_e}{e} (u_e - u_i)$$

(7.156)

where we have used the relations 7.148 for $\nu_{\text{el}}$ and $\text{g}_T$. It can be seen that Eq. 7.156 determines exclusively the relative electron and ion velocity. For the ambipolar regime ($u_e = u_i$) Eq. 7.156 yields the strength of the electric field ensuring ambipolarity:

$$E_A = -\frac{\text{g}_T}{e} \frac{\operatorname{grad} T_e}{T_e} = -0.71 \frac{\operatorname{grad} T_e}{T_e}$$

(7.157)

Representing in the general case the field as the sum of the ambipolar field and the current-inducing field $E_p$, we find from Eq. 7.156 the conductivity of a fully ionized plasma in the field $E_p$:

$$j = -ne(u_e - u_i) = 2 \frac{ne^2}{m_e \tilde{\nu}_e} E_p; \quad \sigma = \frac{2ne^2}{m_e \tilde{\nu}_e}$$

(7.158)

The conductivity equation coincides with Eq. 7.153.

7.10 ENERGY TRANSFER IN HIGHLY IONIZED PLASMAS

Let us now find the expression for the electron heat flux in a highly ionized plasma. The stationary heat flux equation for the case where the frequency of electron–atom collisions is velocity independent can be obtained by substituting the collision term 6.80 into Eqs. 7.151 and 6.80. It has the form

$$\frac{5nT_e}{2m_e} \operatorname{grad} T_e = -(\nu_{\text{el}} + 1.87\tilde{\nu}_e)q_e + 3 \frac{\tilde{\nu}_e nT_e}{2m_e} (u_e - u_i)$$

(7.159)

where the first term on the right-hand side covers the electron–atom, electron-ion, and electron-electron collisions. Equation 7.159 defines the electron heat flux

$$q_e = -\mathcal{K}_e \operatorname{grad} T_e + \text{g}_T nT_e (u_e - u_i)$$

(7.160)

where

$$\mathcal{K}_e = \frac{5}{2m_e (\nu_{\text{el}} + 1.87\tilde{\nu}_e)^2}; \quad \text{g}_T = \frac{3}{2} \frac{\tilde{\nu}_e}{\nu_{\text{el}} + 1.87\tilde{\nu}_e}$$
(because the kinetic coefficients are symmetrical, the coefficient \( g_T \) coincides with the coefficient appearing in the thermal force expression 7.147). The expression obtained differs from the corresponding equation for a weakly ionized plasma (see Eq. 7.22) in that the collision frequency \( \nu_{ei} \) is replaced by the sum frequency \( \nu_{ei} + 1.87 \nu_{ed} \) in the coefficient of thermal conductivity and that a term appears which is proportional to the directed velocity (it is associated with the velocity dependence of the collision frequency \( \nu_{ei} \)).

At a sufficiently high degree of ionization, when \( \nu_{ei} \gg \nu_{ed} \), the thermal conductivity coefficient takes the form

\[
\kappa_e = \frac{5}{2} \frac{g_{a} \rho}{m_e \nu_{ei}} \frac{n T_e}{8 \sqrt{2 \pi} e^{m_e^* L_e}} \]  

(7.161)

where \( g_{a} = 0.54 \). The thermal conductivity coefficient strongly depends on the electron temperature \( \langle T_e^{1/2} \rangle \) and is practically independent of the density (as in the conductivity equation 7.153, there is no such dependence because the frequency of electron–ion collisions \( \nu_{ei} \) is proportional to the density).

A more accurate calculation with the inclusion of the higher moments of the distribution function leads to a considerable change of the numerical coefficients appearing in Eqs. 7.160 and 7.161. For a fully ionized plasma it yields a coefficient \( g_{a} = 1.22 \). Note that this value is obtained by considering both the electron–ion and electron–electron collisions. When only electron–ion collisions are taken into account, the coefficient \( g_{a} \) is much larger (\( g_{a} = 5 \)). The value of electron thermal conductivity is high because the energy is principally transported by the fast electrons, which collide with ions more seldom than the slow. Electron–electron collisions greatly reduce the difference between the contribution of the fast and slow electrons to the transport, since they result in effective energy exchange between them.

The thermal flux of ions and atoms can be obtained from Eq. 6.80 with the collision terms 6.87 and 6.88. We give here the expression for the thermal ion flux in a plasma with a high degree of ionization, in which the frequency of ion–ion collisions is much higher than for ion–atom collisions, \( \nu_{ii} \gg \nu_{ei} \). The collision term of the equation of the thermal flux of ions is practically determined by the ion–ion collisions alone (ion–electron interaction plays a much lesser role because of the low electron mass). Accordingly, the thermal flux is equal to

\[
q_i = -\kappa_{ei} \nabla T_i 
\]

\[
\kappa_{ei} = \frac{5}{2} \frac{g_{a} \rho}{m_i \nu_{ii}} \frac{n T_i}{8 \sqrt{2 \pi} e^{m_i^* L_i}} \frac{T_i^{1/2}}{T_i^{1/2}} 
\]

(7.162)

The use of the collision term 6.85 in the thermal flux equation 6.80 leads to \( g_{ii} = 1.25 \). A more accurate calculation yields \( g_{ii} = 1.56 \). Note that at similar temperatures \( T_e \) and \( T_i \) the coefficient of ion thermal conductivity (Eq. 7.162) is much less than for the electrons (Eq. 7.161):

\[
\frac{\kappa_{ei}}{\kappa_e} \approx \left( \frac{T_i}{T_e} \right)^{1/2} \left( \frac{m_i}{m_e} \right)^{1/2} 
\]

(7.163)

The expressions obtained for the heat fluxes make it possible to establish the equations of balance of the average energies of the charged particles in a highly ionized plasma. At a sufficiently high degree of ionization, when \( \nu_{ii} \gg \nu_{ei} \) and \( \nu_{ii} \gg \nu_{ii} \), these equations result from substitution of Eqs. 7.161 and 7.162 into Eq. 6.79. Without an external electric field (at \( u_i = u_e = 0 \) they take the following form:

\[
\frac{n \partial T_e}{\partial t} = \frac{5}{3} \frac{g_{a} \rho}{m_e \nu_{ei}} \frac{n T_e}{m_e \nu_{ei}} \nabla T_e - \kappa_e \nu_{ei} \nabla (T_e - T_i) 
\]

(7.164)

\[
\frac{n \partial T_i}{\partial t} = \frac{5}{3} \frac{g_{a} \rho}{m_i \nu_{ii}} \frac{n T_i}{m_i \nu_{ii}} \nabla T_i - \kappa_{ei} \nu_{ei} \nabla (T_e - T_i) - \frac{1}{2} \nu_{ii} n (T_i - T_a) 
\]

(7.165)

where it is taken into consideration that the coefficient of energy transfer on collisions of ions and atoms \( \kappa_{ei} = \frac{1}{2} \), since \( m_i = m_a \); the energy transfer coefficient for elastic electron–ion collisions \( \kappa_{ei} = 2 m_i/m_e \); and, in principle, \( \kappa_{ei} \) can also cover the effect of inelastic losses of electron energy. The first term on the right-hand side of the equations describes the heat transfer. The second term determines the energy exchange between the electrons and ions. The third term on the right-hand side of the ion equation determines the energy losses on collisions of ions with neutral particles; these collisions can be substantial even in a highly ionized plasma, since \( \kappa_{ei} = \frac{1}{2} \), while \( \kappa_{ei} = 2 m_i/m_e \ll 1 \).

To evaluate the relative role of the different terms we can introduce characteristic times determining the efficiency of the corresponding processes \( \tau_{eq} = T_e/(\partial T_e/\partial t) \); the heat transfer times for electrons and ions,

\[
\tau_{eq} = \frac{n L^2_T}{\kappa_e} = \frac{m_i \nu_{ii}}{T_i} L_i^2; \quad \tau_{eq} = \frac{n L^2_T}{\kappa_i} = \frac{m_e \nu_{ei}}{T_i} L_i^2 
\]

(7.166)

(here \( L_T := [(1/T) \partial T/\partial t] \)), and the energy exchange times,

\[
\tau_{ei} = \frac{1}{\kappa_e \nu_{ei}}; \quad \tau_{ei} = 2 \frac{1}{\kappa_{ei} \nu_{ei}} 
\]

For a fully ionized plasma we can derive the summary heat transfer equation by adding up Eqs. 7.164. Neglecting the thermal conductivity,
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since \( \mathcal{H}_e \ll \mathcal{H}_p \) (Eq. 7.163), and assuming that the inelastic electron energy losses are insignificant \( \kappa_d = \kappa_a = 2m_e/m_i \), we have
\[
n = \frac{n}{3} \left\{ \frac{1}{T_e} \right\} \frac{5}{3} g_e \frac{\text{div} \left( \frac{n \nabla T_e}{m \nu_e} \right)}{\text{grad} \ T_e} = 0 \quad (7.167)
\]

The ratio between the electron and ion temperatures depends on the efficiency of energy exchange between the electrons and ions. With a relatively high efficiency, when \( \tau_e \ll \tau_m \), it can be assumed that \( T_e = T_i \). In the opposite case the ratio depends on the initial and boundary conditions and can be obtained from the electron equation of energy balance.

### 7.11 ELECTRON "RUNAWAYS"

In a highly ionized plasma, where the frequency of collisions of electrons with ions greatly exceeds that of their collisions with neutral atoms \( \nu_e \gg \nu_{ea} \), the mean free path of the electrons rapidly increases with energy. In a sufficiently strong electric field, where the electrons accumulate, between collisions, an energy comparable with the random, they can go into the continuous-acceleration regime. This transition is called the effect of runaway of electrons (they "run away" from collisions).

The conditions for transition of electrons to the runaway regime can be evaluated with the aid of the averaged equation of motion of electrons in an electric field. In accordance with this equation the stationary value of the directed velocity can be found by equating the electric force and the friction force due to the electron-ion collisions:
\[
eE = R_d = -m_e \nu_e (v_e v) \quad (7.168)
\]

where
\[
\nu_e = \frac{4 \pi n m_e e}{m_e v^3} I_e \approx \beta \frac{n}{v^3} \quad (7.169)
\]

Here \( v \) is the relative velocity, which is practically equal to the electron velocity, and \( \langle \cdot \rangle \) means averaging over the velocities. For a weak electric field, in which the directed electron velocity is much less than the random one \( u_e \ll w \), the averaging results in an equation corresponding to Eq. 7.146:
\[
R_d = -0.51 m_e \bar{v}_e u_e = -0.14 \beta m_e n \left( \frac{m_e}{T_e} \right)^{5/2} u_e \quad (7.170)
\]

Here the friction force \( R_d \) increases linearly with \( u_e \).

For a strong field in which the energy accumulated by the electrons between collisions is much higher than the thermal energy \( u_e \gg w \), \( \nu_e \approx u_e \) and the collision frequency is practically independent of the random velocity. In this case the friction force
\[
R_d = -m_e u_e \nu_e (u_e) \approx \frac{\beta m_e n u_e}{u^3} \quad (7.171)
\]

is inversely proportional to the square of the directed velocity. Thus the dependence of \( R_d \) on \( u_e \) represents a curve with a maximum at \( u = w \). To find its approximate estimates, we can use the following equation, which is correct to within a numerical factor of the order of unity:
\[
R_d = -\frac{\beta m_e n u_e}{(u^2 + v_e^2)^{3/2}}, \quad v_e^2 = \frac{3T_e}{m_e} \quad (7.172)
\]

A more accurate calculation of \( R_d \) at \( u = w \) can be accomplished by averaging Eq. 7.168 over the distribution function. The result of the averaging carried out, with the assumption that the random velocity distribution is Maxwellian, is presented in Fig. 7.7. From Eq. 7.172 it is seen that the maximum value of the friction force at \( u = v_e \) is equal to
\[
R_{max} = 0.2 \beta \frac{m_e n}{T_e} = 0.2 \frac{4 \pi n e^4}{T_e} L_e
\]

\[
= 0.75 m_e^{1/2} T_e^{1/2} v_e \quad (7.173)
\]

At \( |eE| > R_{max} \) the friction force cannot offset the electric force at any directed velocity. Under the effect of the electric field the electrons will go into the continuous-acceleration regime. The critical field strength, which determines the transition boundary, is
*\[ E_c = 0.2 \frac{4 \pi n e^4}{T_e} L_e = 0.2 \frac{eE_{cr}}{r_b} \quad (7.174) \]*

It is easy to ascertain that the average energy accumulated by the electrons in such a field along the mean free path is of the order of the average friction force on runaway electrons as a function of energy.

*The condition 7.174 is sometimes called the Dreicer regime.*
thermal energy

\[ W_\text{th} = eE \lambda_d \approx \left( \frac{eE}{\lambda_d} \right) \sqrt{\frac{T_e}{m_e}} \approx T_e \]

Although at typical plasma parameters the critical field (Eq. 7.174) is not high, it is usually difficult to set up such a field in a fully ionized plasma possessing a high conductivity. But at lower field strengths “partial runaway” of electrons is also possible; that is, transition to the acceleration regime of fast electrons, for which the collision frequency, determining the friction, is lower than for the bulk of the electrons.

Let us consider, for instance, a group of electrons moving in the direction of the external electric force with similar velocities substantially exceeding the thermal velocities \( v_p \approx v_T \). The friction force acting on these electrons is due to their collisions with ions and with the bulk of the electrons. We can readily estimate this force by assuming that the relative velocity on collisions is equal to the velocity of the particles of the isolated group:

\[ R(v_p) = R_\text{el}(v_p) + R_\text{ion}(v_p) \approx \mu_{de} v'_e(v_p) v_p + \mu_{de}^2 v'_\text{el}(v_p) v_p \quad (7.175) \]

This estimate neglects collisions with electrons whose velocity exceeds \( v_p \), but at \( v_p \approx v_T \) the number of such electrons is small. The electrons of the isolated group will be accelerated if the electric force acting on them is larger than the friction force. Using Eq. 2.69 for electron collision frequencies, we write this condition as

\[ eE > m_e v_p \left[ v'_e(v_p) + \frac{1}{2} v'_\text{el}(v_p) \right] = \frac{12\pi e^4}{m_e v_p^2} L_e n \]

which takes into account that \( \mu_{de} = m_e \) and \( \mu_{de} = m_i / 2 \). With a given electric field it determines the velocity from which electrons moving in the direction of the field are accelerated

\[ \frac{m_e v_p^2}{2} > \frac{12\pi e^4 L_e}{ET_e} = \frac{15E}{E} \quad (7.176) \]

Here \( E_\text{c} \) is the critical field strength (see Eq. 7.174). Thus at \( E < E_\text{c} \), the bulk of the electrons in the electric field move with a quasi-stationary directed velocity, whereas the electrons from the “tail end” of the distribution function find themselves in the runaway regime. In this region of the velocity space the distribution function is no longer at equilibrium. With time, the electrons acquire an ever-increasing energy, and the distribution function is “stretched” along the axis in the direction of the electric force. This process reduces the distribution function below the equilibrium near the runaway boundary. As a result there arises an excess diffusion flux of electrons from the low (thermal)—velocity region to the runaway boundary, which is due mainly to electron–electron collisions. If there were no obstacles to unlimited acceleration, all the plasma electrons would gradually find themselves in the runaway regime. Actually, however, a number of limitations hinder acceleration. In the first place the acceleration time is limited because of fast electron runaway from the plasma volume. Further, in a nonfully ionized plasma, acceleration is limited by electron–neutral particle collisions. In the presence of complex ions a limitation is imposed because at a small distance the ion potential differs considerably from the coulomb, and at high velocities the collision frequency ceases to reduce.

Another factor that may hinder acceleration is electron emission during the motion in the plasma (“radiation friction”). Finally, mention should be made of the possible kinetic instabilities during the passage of fast electrons through the plasma.

We discuss only the limitation of acceleration associated with collisions of electrons with neutral particles. Allowing for these, the friction force acting on the electron gas as a whole is equal to

\[ R = R_\text{el} + R_\text{ion} = m_e v'_e(v_p) v_p + m_e v'_\text{el} v_p \]

where we assumed that \( v'_\text{el} \) is velocity independent. For the group of fast electrons moving in the direction of \( E \) with the isolated velocity \( v_p \), we obtain the friction force by adding \( R_\text{el} \) to Eq. 7.175:

\[ R(v_p) = R_\text{el} + R_\text{el} + R_\text{ion} = \frac{12\pi e^4}{m_e v_p^2} + m_e v'_\text{el} v_p \quad (7.178) \]

The dependence of the values of \( R \) on \( u \) and \( v_p \) is illustrated in Fig. 7.8. It can be seen that the curves \( R(u) \) have a minimum at \( u > v_p \) and \( R(v_p) \). The minimum of \( R(v_p) \) corresponds to \( v_m = 2(v_p + v_a) \) and is equal to

\[ R_{\text{min}} = m_e v_m v_p = (9\sqrt{2\pi} m_e T_e v_p^2)^{1/3} \]

In electric fields where the electron-accelerating force is less than the minimum friction force \( eE < R_{\text{min}}(v_p) \) electron runaway is clearly impossible. The value of the threshold field characterizing the runaway boundary at \( v_{\text{as}} \approx v_a \) can be found from the relation

\[ \frac{E_\text{c}}{E} = \frac{R_{\text{min}}(v_p)}{R_{\text{max}}} \approx 4 \left( \frac{v_{\text{as}}}{v_p} \right)^{2/3} \quad (7.179) \]

In fields less than \( E_\text{c} \) the stationary electron velocity distribution may be...
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Fig. 7.8 Dependence of friction force on energy and velocity.

near-Maxwellian. In high fields $E_c < E < E_r$, there is a velocity region within which the electrons must be entrained in the acceleration regime (see Fig. 7.8). This region, however, is bounded on the high-velocity side by the condition of equality of the electric force to the force of friction against the neutral particles

$$m_e v_{e0} u = eE$$

Near the boundary of the acceleration region, at $v = u$, electron accumulation must take place; a distribution function maximum must then form, whose width depends on the ratio between the electron acceleration in the electric field and the diffusion in the velocity space due to electron-electron collisions. At $E > E_r$, when the bulk of the electrons go into acceleration, the entire distribution must shift into the velocity field in which the electric force is offset by the friction of the electrons against the neutral gas.

We now consider another peculiarity in the behavior of a highly ionized plasma in an electric field, which is sometimes called the electron "energy runaway." This may occur when the principal mechanism of energy losses of the electrons is their cooling on collisions with ions. The stationary electron energy balance equation (Eq. 6.79) in this case amounts to the equilibrium between the energy acquired by the electrons in the electric field $P_E$ and the energy lost by them on collisions with ions $P_d$:

$$P_E = \sigma E^2 = \frac{2\pi e^2 E^2}{m_e v_{e0}}; \quad P_d = \frac{3}{2} k_e n v_a (T_e - T_i)$$

(7.180)

Recalling that $v_a(T_e) \sim T_e^{-3/2}$, we reduce the energy balance equation to

$$\frac{4e^2E^2}{3} \frac{1}{m_e} = \frac{k_e n v_a(T_e)^2 T_e}{T_i^2} \left( \frac{T_e}{T_i} - 1 \right)$$

(7.181)

It is easy to see that at $k_e = \text{const}$ the dependence of the right-hand side of the equality on $T_e$ has a maximum at $T_e = 1.5 T_i$. Obviously, the stationary electric field in the case at hand cannot exceed the value corresponding to this maximum:

$$E_k = 0.4 \sqrt{\frac{k_e n m_i T_i}{e} v_a(T_i)} = 0.53 \sqrt{\frac{k_e E_c(T_i)}{e}}$$

(7.182)

where $E_c(T_i)$ is the critical field at $T_e = T_i$ (see Eq. 7.174). The dependence $T_e(E)$, determined by Eq. 7.181, is illustrated in Fig. 7.9. It is seen that a gradual increase in field strength $E$ from zero to $E_k$ raises the stationary electron temperature from $T_e$ to $1.5 T_i$. With a further increase in field the stationary solution of the energy balance equation vanishes. This means that the energy accumulated by the electrons in the electric field exceeds the energy losses on collisions, and the electron temperature rises with time. The temperature growth leads to a further drop in collision frequency $T_e$ and to a greater "unbalance." Thus the field $E_k$ yields the boundary of transition to the regime of increasing heating (energy runaway).

Fig. 7.9 Dependence of electron temperature on electric field.
Under actual conditions the energy runaway effect is limited by other loss mechanisms. In a nonfully ionized plasma the losses due to electron–neutral particle collisions may be considerable. Their inclusion amounts to adding the quantity \( P' = (\kappa_{\text{ne}} n H = (T - T_a) \) on the right-hand side of the balance equation (Eq. 7.180). At \( \kappa_{\text{ne}} n H = (T - T_a) \) these losses are relatively small in the temperature range of \( T_e \sim T_i \). But with increasing temperature at \( T_e \sim T_i \) they become predominant, because \( v_{\text{ne}} \sim 1/T_e^{3/2} \), while \( v_{\text{ne}} = \text{const} \). Accordingly, on the curve \( E(T_e) \) there appears, after the maximum, a growing branch determined by electron–atom collisions (see Fig. 7.9). Its position can be found from the balance condition

\[
P' \approx P_n; \quad T_e \approx \frac{2e^2 E^2}{m_e \kappa_{\text{ne}} n H (T_e)}
\] (7.183)

In this case, as can be seen from Fig. 7.9, the increase in electric field from zero to \( E \), which results in transition to the nonstationary heating regime, terminates with a jump to the region of \( T_e \gtrsim T_i \). Note that the reverse jump with a decrease in field occurs at a lower value of the field than the forward jump, that is, a kind of a heating hysteresis is observed.

Another essential limitation of the energy runaway effect can be imposed by the thermal conductivity of the plasma, since the thermal conductivity coefficient increases rapidly with electron temperature, \( \kappa_{\text{th}} \sim T_e^{3/2} \). A quantitative consideration of this limitation requires the solution of the nonlinear balance equation, and here we restrict ourselves to a crude estimate. It can be obtained by equating the value of the energy accumulated by the electrons in the field \( P_e \) (Eq. 7.180) to the average energy losses due to thermal conductivity:

\[
P_e = \text{div}(\kappa_{\text{th}} \text{grad} T_e) \sim \frac{\kappa_{\text{th}} T_e^3}{L_e^3} \approx \frac{n T_e^3}{m_e e L_e}
\]

where \( L_e = [(1/T_e) \text{grad} T_e]^{-1} \). As a result of the estimate we find the temperature in the electric field where the thermal conductivity is the principal source of losses:

\[
T_e \approx e E L_e
\] (7.184)

8

MOTION OF CHARGED PLASMA PARTICLES IN MAGNETIC FIELD

8.1 SOME DATA ON STATIC MAGNETIC FIELDS

The effect of a magnetic field on the charged particle motion determines its spatial distribution. Therefore we first discuss some characteristics of the spatial properties of a magnetic field.

It is common knowledge that the strength distribution of the magnetic field \( \mathbf{H}(r) \) is described by the equations

\[
\text{curl} \mathbf{H} = \frac{4\pi}{c} j
\]

\[
\text{div} \mathbf{H} = 0
\]

With given current distribution and boundary conditions these equations determine the field \( \mathbf{H}(r) \) unambiguously. Field variation in space is characterized by a vector derivative \( \mathbf{H} \), which is a tensor. In Cartesian coordinates it has the form

\[
\nabla \mathbf{H} = \left[ \begin{array}{c} \frac{\partial H_x}{\partial x}, \frac{\partial H_x}{\partial y}, \frac{\partial H_x}{\partial z} \\
\frac{\partial H_y}{\partial x}, \frac{\partial H_y}{\partial y}, \frac{\partial H_y}{\partial z} \\
\frac{\partial H_z}{\partial x}, \frac{\partial H_z}{\partial y}, \frac{\partial H_z}{\partial z} \end{array} \right]
\]

The tensor components are related by Eqs. 8.1 and 8.2. In particular, the sum of the diagonal terms, which is equal to div \( \mathbf{H} \), is obviously zero. In current-free areas \( \text{curl} \mathbf{H} = 0 \) and

\[
\frac{\partial H_x}{\partial x} = \frac{\partial H_y}{\partial y} = \frac{\partial H_z}{\partial z}
\]

In describing the spatial behavior of \( \mathbf{H} \) the line-of-force concept is often used. By a magnetic line of force is meant some imaginary spatial