5 DISTRIBUTION FUNCTION OF CHARGED PARTICLES IN ELECTRIC FIELD

5.1 EFFECT OF ELECTRIC FIELD ON CHARGED PARTICLE VELOCITY DISTRIBUTION

Under the effect of external forces particle distribution functions deviate from equilibrium. Not only can the average energy and directed velocity of the particles change in such cases, but also the type of the distribution function. The effect of the forces on the distribution function is different for different species of particles. In the electric field charged particles are accelerated, changing their energy. The magnetic field changes their trajectory. Neutral particles remain completely unaffected by these forces. The neutral particles of a plasma are usually in close contact with the environment (for instance, with the chamber walls). Therefore the average energy of the neutral particles is lower than that of the other components, and their velocity distribution is closer to equilibrium.

Let us determine the conditions under which an electric field substantially affects the charged particle velocity distribution in a weakly ionized plasma. Accelerating in the electric field, the charged particles acquire an additional velocity within the intercollisional time $\tau$:

$$\Delta v_E = \frac{eE}{m} \tau = \frac{eE}{m v} \tau$$  \hspace{1cm} (5.1)

The corresponding energy increment is equal to

$$\Delta K_E = \frac{m (v + \Delta v_E)^2}{2} - \frac{m v^2}{2} = m v \Delta v_E + \frac{m (\Delta v_E)^2}{2}$$

or, after averaging (at $\langle v \rangle = 0$),

$$\langle \Delta K_E \rangle = \frac{m (\Delta v_E)^2}{2} = \frac{e^2 E^2}{2 m v^2}$$  \hspace{1cm} (5.2)

At the same time the energy losses of charged particles in collisions with neutral ones are proportional to the difference of their energies (see Section 2.1):

$$\Delta K_e = \kappa (K - K_e)$$  \hspace{1cm} (5.3)

In a weakly ionized plasma these losses usually predominate. Therefore in the stationary state the average energy acquired in the field must be equal to that lost in collisions $\langle \Delta K_E \rangle = \langle \Delta K_e \rangle$. The energy balance determines the difference between the average energies of the charged and neutral particles:

$$\langle K - K_a \rangle = \frac{e^4 E^2}{m k T}$$  \hspace{1cm} (5.4)

where $\nu$ and $\kappa$ are the averaged values of the collision frequency and of the energy transfer coefficient, respectively.

With the aid of Eq. 5.4 we obtain the condition for a small electric field effect on the average energy $\langle K - K_a \rangle \ll \langle K \rangle = T$, or

$$E \ll E_p = \frac{\nu}{e} \frac{\sqrt{m T}}{\sqrt{\kappa T}} = \frac{\sqrt{\kappa T}}{e \lambda}$$  \hspace{1cm} (5.5)

where $\lambda = v_0/v = (1/\nu)(1/\sqrt{T/m})$ is the mean free path. If this condition fails, that is, if $E \gg E_p$, the average energy of the charged particles considerably exceeds that of the neutral particles in accordance with Eq. 5.4. At $E \gg E_p$ the average random velocity can be obtained from the relation

$$v_T = \sqrt{2(K)/m} = \sqrt{(1/\nu)kT}$$  \hspace{1cm} (5.6)

The average directed velocity $u_E$ is approximately equal to the averaged velocity increment within the intercollisional time (Eq. 5.1), since collisions result in a substantial change in velocity direction. From this we find

$$u_E = \langle \Delta v_E \rangle = \frac{eE}{m v} = v_T \sqrt{\kappa}$$  \hspace{1cm} (5.7)

For ions the energy transfer coefficient in elastic collisions is close to unity ($\kappa = m_i/m_a \approx 1$). Here the condition 5.5 becomes $E \ll E_p = T/e \lambda_{in}$ or, in numerical form, $E (V/cm) \ll 10^2 T_2 (eV) p (mm Hg)$. (Here we used...
the average value $\lambda_0 \approx 10^{-2}\mu$; the values of $\lambda_0$ for various gases do not differ greatly from it. If the condition is not satisfied, the average energy and the distribution function deviate considerably from equilibrium. Then, for ions, $u_{\text{dir}} = v_{\text{r}}$ in accordance with Eq. 5.7; that is, the distribution function is essentially anisotropic.

For electrons the energy transfer coefficient is usually much less than unity; in elastic collisions $\kappa = 2m_e/m_n$. Therefore the criterion of a weak effect of the electric field on the distribution function 5.5 is much more rigid than for the ions $E = E_{\text{eq}} = \sqrt{\kappa m_e T_e}/\epsilon_m$. For electrons in hydrogen, for instance, $E_{\text{p}} = T_e/\mu$ at $K < 2 \text{ eV}$ and $E_{\text{p}} \approx 1.5 T_e^{1/2}/\mu$ at $K > 2 \text{ eV}$, whereas in neon $E_{\text{p}} \approx 5 \times 10^{-3} T_e/\mu$ (here $E_{\text{p}}$ is given in V/cm, $T_e$ in eV, and $\mu$ in mm Hg). Hence the average electron energy depends on the electric field, even if it is comparatively weak. But irrespective of the field value the directed velocity 5.7 is much less than the random velocity $u_{\text{dir}} / v_{\text{r}} \approx \sqrt{\kappa}$; that is, the anisotropy of the distribution function is low.

As has already been noted, the directed velocity is relatively low because in each collision an electron sharply changes its direction, whereas the velocity and energy change vary little ($\Delta K \approx K\kappa$); that is, the electron “accumulates” its energy during many intercollisional periods. In other words, the momentum relaxation time is determined by the intercollisional time $(\tau_\rho \approx 1/\nu_{\text{dir}})$, whereas the energy relaxation time greatly exceeds it $(\tau_\kappa \approx 1/\nu_{\text{dir}})$. In a highly ionized plasma, when considering the energy balance of the electrons one must take into account their collisions not only with neutral particles, but also with ions. The condition for a weak effect of the electric field on the average electron energy can, as before, be represented in the form of the inequality 5.5 by using the total frequency of collisions of electrons with atoms and ions $\nu_{\text{dir}} = \nu_{\text{dir,at}} + \nu_{\text{dir,ion}}$. Note that as the electric field strength approaches its critical value (Eq. 5.5) the heating of the electrons reduces the frequency of their collisions with ions $(\nu_{\text{dir}} \approx 1/T_\text{e}^{3/2})$. The reduction of $\nu_{\text{dir}}$ in turn, increases the energy received by the electrons from the field (Eq. 5.2) and decreases the energy losses (Eq. 5.3); that is, it results in a further rise of the electron temperature. This process is limited either by collisions of electrons with neutral particles or by other types of losses, such as those due to radiation or thermal conduction. In the first case the conclusion about the weak anisotropy of the distribution function remains valid.

### 5.2 Method for Solving Kinetic Equation

When the deviation of the distribution function from equilibrium is small, the solution of the kinetic equation can be found by the method of successive approximations. When using this method the distribution function is represented as a series in the powers of the parameters determining its deviation from equilibrium (forces affecting the particles and the concentration and temperature gradients):

$$ f = f_{(0)} + f_{(1)} + f_{(2)} + \cdots $$

(5.8)

The first term of the series $f_{(0)}$ is the equilibrium (Maxwellian) distribution, the second term $f_{(1)}$ includes a linear combination of the parameters, the third $f_{(2)}$ a quadratic combination, and so on. Substituting the series into the kinetic equation yields a sequence of equations of different orders of smallness. The first-approximation equation includes functions $f_{(0)}$ and $f_{(1)}$ and the second-approximation equation $f_{(0)}, f_{(1)}, f_{(2)}$, and so on. Solving them successively, we can find corrections of different orders to the distribution function. The set of these corrections describes both the deviation of the spherically symmetric part of the distribution function from the Maxwellian distribution and the anisotropy of the distribution function owing to deviations from equilibrium. Determination of the anisotropic component is particularly important in this case of small deviations from the equilibrium distribution, since it enables one to calculate such important macroscopic characteristics as the directed velocity, the energy flux, the momentum flux, and so on. We do not dwell on these calculations; in Chapter 6 we describe a method for obtaining anisotropic characteristics from the moments equation.

When the deviation of the distribution function from equilibrium is large, the method for solving the kinetic equation that uses expansion in the powers of perturbation is inapplicable. But for electrons in an electric field one can use another method of expansion based on the low anisotropy of the distribution function. As demonstrated in Section 5.1, the directed velocity is usually much lower than the random because of the small losses of electron energy in collision with heavy particles. Therefore, even in a strong electric field, with deviations from equilibrium the anisotropy of the electron velocity distribution remains low. Owing to this, when solving the kinetic equation one can use an expansion of the distribution function in parameters characterizing its anisotropy. The rapid convergence of the series permits us to restrict ourselves to a small number of terms and find rather easily both the anisotropy and the symmetric part of the distribution function.

Let us consider in more detail the case where a homogeneous electric field $E$ is the source of disequilibrium. Here the field determines the only isolated direction (we assume that the $0z$ axis is parallel to it). Accordingly, the electron velocity distribution function may depend only on the velocity $v$ and on the angle $\Theta$ between the directions of velocity $v$ and
field $E$. The dependence on the angle $\Theta$, which is due to the anisotropy of the distribution function, must be low. Therefore it is natural to represent this dependence as an expansion in orthogonal Legendre polynomials $P_n(\cos \Theta)$:

$$f(v) = \sum_{n=0}^\infty f_n(v)P_n(\cos \Theta) \tag{5.9}$$

where the functions $f_n$ depend only on the velocity. By substituting the expansion 5.9 into the kinetic equation it is easy to get a system of coupled equations for the functions $f_n(v)$. Each of them is obtained by multiplying the kinetic equation by one of the polynomials $P_n(\cos \Theta)$ and integrating over all the values of $\cos \Theta$. The first two terms of the sum in Eq. 5.9 contain the polynomials $P_0 = 1$ and $P_1 = \cos \Theta$. With small anisotropy we can restrict ourselves to these terms in solving many problems. Then we have

$$f(v) = f_0(v) + \cos \Theta f_1(v) = f_0(v) + \frac{v}{v} f_1(v) \tag{5.10}$$

It is easy to see that the function $f_0(v)$ defines the average electron energy and the average value of any other energy-dependent quantity. Indeed, by averaging $g(v)$ with the aid of Eq. 3.8 we obtain

$$\langle g(v) \rangle = \int_0^\infty g(v)f_0(v) \, dv = \int \int \int g(v)[f_0(v) + f_1(v) \cos \Theta]v^2 \sin \Theta \, dv \, d\Theta \, d\phi$$

$$= 4\pi \int_0^\infty g(v)f_0(v)v^2 \, dv \tag{5.11}$$

(here we changed to spherical coordinates $v, \Theta, \phi$ in the integrand).

From Eq. 5.11 we can see the relationship between $f_0$ and the distribution functions with respect to the total velocities $f_i$ (Eq. 3.6) and energies $f_i$ (Eq. 3.7):

$$f_0(v) = 4\pi v^2 f_0(v); \quad f_i(K) = \frac{4\sqrt{2\pi}}{m^{3/2}} \sqrt{K} f_0 \left( \frac{\sqrt{2K}}{m} \right)$$

This relationship also determines the normalization condition

$$\int_0^\infty 4\pi v^2 f_0(v) \, dv = 1 \tag{5.12}$$

In a similar manner we can ascertain that the average electron velocity is defined by the function $f_1(v)$. For the velocity component parallel to the electric field we get

$$u_x = \langle v \cos \Theta \rangle = \int \int \int v \cos \Theta[f_0(v) + f_1(v) \cos \Theta]v^2 \sin \Theta \, dv \, d\Theta \, d\phi$$

$$= 4\pi \int_0^\infty v^3 f_1(v) \, dv \tag{5.13}$$

Therefore it is natural to define the function $f_1(v)$ as

$$f_1(v) = \frac{1}{4\pi} \int \int \int v^2 \sin \Theta \, dv \, d\Theta \, d\phi$$

Accordingly, the function $f_0(v)$ is called the isotropic component, and the function $f_1(v)$ the directed component of the distribution function.

In order to obtain the equation for $f_0(v)$ and $f_1(v)$ we must substitute the sum of Eq. 5.10 into the kinetic equation. For a homogeneous plasma in a homogeneous electric field the kinetic equation 3.17 can be written as

$$\frac{\partial f}{\partial t} - \frac{e}{m} E \frac{\partial f}{\partial v} = \frac{\partial}{\partial t}\left(\frac{1}{m} f \frac{e}{m} E \frac{\partial f}{\partial v} + \frac{e}{m} E \frac{\partial f}{\partial \cos \Theta} \cos \Theta \frac{\partial f}{\partial v} \right)$$

where $f(x) = f_0 + f_1 \cos \Theta$. Multiplying the kinetic equation by $\cos \Theta$ and integrating over all the values of $\cos \Theta$ from $-1$ to $1$ as a result of integrating the terms proportional to $\cos \Theta$ vanish in Eq. 5.15, and the equation takes the form

$$\frac{\partial f_0}{\partial t} + \cos \Theta \frac{\partial f_1}{\partial t} - \frac{e}{m} E \frac{\partial f_0}{\partial v} + \cos \Theta \frac{e}{m} E \frac{\partial f_1}{\partial v} = S_0 \tag{5.16}$$

where the following notation is introduced:

$$S_0 = \frac{1}{2} \int_{-1}^1 \frac{\partial f}{\partial t} \, d(\cos \Theta) \tag{5.17}$$

The second equation is obtained by multiplying Eq. 5.15 by $\cos \Theta$ and integrating again over all the values of $\cos \Theta$. In this case the terms independent of $\Theta$ vanish. Then

$$\frac{\partial f_0}{\partial t} - \frac{e}{m} E \frac{\partial f_0}{\partial v} = S_1 \tag{5.18}$$
where
\[ S_1 = \frac{3}{2} \int \frac{\delta(n_f)}{\delta t} \cos \theta \, d(\cos \theta) \]  
(5.19)

Simultaneous solution of Eqs. 5.16 and 5.18 makes it possible to find both components of the distribution function \( f_d(v) \) and \( f_s(v) \).

A similar method for expanding the electron distribution function can be applied to a more general case, where a magnetic field is present as well as the electric one and the plasma cannot be considered homogeneous. Then the distribution function may depend on all the velocity components. With a weak anisotropy it can be sought, as in Eq. 5.10, in the form of the sum of the isotropic component \( f_0(v) \) and three directed components \( f_{ix}(v) \), which define the components of the average electron velocity:
\[ f(v) = f_d(v) + \frac{v_x}{v} f_{ix}(v) + \frac{v_y}{v} f_{iy}(v) + \frac{v_z}{v} f_{iz}(v) \]
(5.20)

where a vector function \( f_0(v) \) with components \( f_{ix}, f_{iy}, \) and \( f_{iz} \) is introduced. It is easy to see that these functions define the components of the directed velocity. Making use of the representation 5.20, we obtain
\[ u_k = \int v f(v) d^3v = \int \left[ v f_0(v) + \sum \frac{v_{ij}}{v} f_{ij}(v) \right] v^i dv \, d\omega \]
where we switched to spherical coordinates in the integrand \((d\omega = \sin \theta \, d\Theta \, d\phi)\). After integration over the angles the sum retains only the term proportional to \((v^j v^j f_{0})\), and the others vanish. Then we have
\[ u_k = \frac{4\pi}{3} \int v^i f_{ix}(v) \, dv \]  
(5.21)

By substituting the representation 5.20 into the kinetic equation 3.17 we can, as before, find the associated equations for the functions \( f_0, f_{ix}, f_{iy}, \) and \( f_{iz} \). The first one is obtained by averaging the kinetic equation over all the velocity directions (by integrating over the solid angle), and the other three by averaging after multiplying by the direction cosines of the angles between the velocity vector and the coordinate axes. Without dwelling on the calculations we give the resulting equations:
\[ \frac{\partial(n f_0)}{\partial t} + \frac{v}{3} \text{div}(n f_0) - \frac{en}{3m v^2} \frac{\partial}{\partial \theta} (v^2 E \sin \theta) = S_0; \]
\[ \frac{\partial(n f_i)}{\partial t} + v \text{grad}(n f_i) - \frac{en}{m_e} \frac{\partial f_0}{\partial v} - \frac{en}{m_e} (\mathbf{H} \times f_i) = S_i \]
(5.22)

The latter equation is the vector form of the three equations for the components \( f_{ix}, f_{iy}, f_{iz} \).

The collision terms \( S_0 \) and \( S_1 \) in Eq. 5.22 are obtained from the equalities
\[ S_0 = \frac{1}{4\pi} \int \frac{\delta(n_f)}{\delta t} \, d\omega; \]
\[ S_1 = \frac{3}{4\pi} \int \frac{\delta(n_f)}{\delta t} \cos \theta \, d\omega \]  
(5.23)

where \( \Theta_k \) is the angle between the \( k \) axis and the velocity vector, and integration is performed over the solid angles covering all the velocity directions.

### 5.3 COLLISION INTEGRALS FOR ELECTRONS

Let us now determine the collision terms \( S_0 \) and \( S_1 \) appearing in Eqs. 5.16 and 5.18. Each of them can be represented as a sum of integral expressions related to different types of electron collisions:
\[ S_k = \sum_{\ell} S_{k,\ell}^{\alpha} \]  
(5.24)

where the summing is generally done over all the species of \( \beta \)-type particles and embraces elastic (\( \ell \)) and inelastic (\( n \)) collisions with excitation of different \( j \) levels.

We first consider the terms due to elastic collisions of electrons with heavy particles (atoms or ions). For a plasma whose stationary state is maintained by an electric field the average energy of electrons usually greatly exceeds that of the heavy particles. Bearing this in mind, we can use the collision integral for elastic collisions as in Eq. 3.31:
\[ S_k^{(E)} = \frac{n_e}{v^3} \int_{\Omega} d\Omega \left[ v' \sigma(v', \theta) f(v') - v^3 \sigma(v, \theta) f(v) \right] d\Omega \]  
(5.25)

Recall that here \( v' \) is the velocity of an electron before collision, which places it in the velocity range \( v \, dv \) under review. Substituting Eq. 5.25 into Eqs. 5.17 and 5.19 and using the representation of \( f(v) \) as the sum of Eq. 5.10, we obtain the expression for \( S_{0\ell} \) and \( S_{1\ell} \):
\[ S_{0\ell} = \frac{n_e}{2\pi^2} \int_{\Omega} d\Omega \int_{\Omega} d\Omega \left[ v' \sigma(v', \theta) f_0 + f_1 \cos \Theta - v^3 \sigma(f_0 + f_1, \cos \Theta) \right]; \]
\[ S_{1\ell} = \frac{3}{2} \frac{n_e}{v^3} \int_{\Omega} d\Omega \int_{\Omega} d\Omega \left[ v^3 \sigma(f_0 + f_1, \cos \Theta) \right. \]
(5.26)
in which the primed quantities $\sigma$, $f_0$, and $f_1$ are determined by the electron velocity $v'$. The relationship between the angles $\Theta'$ (between the vectors $v'$ and $E$) and $\Theta$ (between $v$ and $E$) can be established using a spherical triangle formed by the vectors $v$, $v'$, and $E$ (Fig. 5.1). From the well-known formula of spherical trigonometry we find 

$$\cos \Theta' = \cos \Theta \cos \vartheta - \sin \Theta \sin \vartheta \cos \varphi,$$

where $\vartheta$ and $\varphi$ are the angles determining the solid scattering angle. Substituting this relation into Eq. 5.26 and integrating with respect to $\Theta$, we get

$$S_0 = \frac{n_m m_e}{v^2} \int_{(0)} (v^' \sigma f_0 - v^4 \sigma f_0) d\Omega;$$

$$S_1 = \frac{n_m m_e}{v^2} \int_{(0)} (v^' \sigma f_1 \cos \vartheta - v^4 \sigma f_0) d\Omega$$

(5.27)

where it is taken into account that on integrating with respect to $d\Omega = \sin \Theta d\Theta d\varphi$, the terms proportional to $\cos \varphi$ reduce to zero.

In elastic collisions with heavy particles the electron velocity change is very small (of the order of the mass ratio). Neglecting it, that is, assuming $v' = v$ and $\sigma' = \sigma$, $f_0 = f_0$, $f_1 = f_1$, we obtain the following expression for $S_1$:

$$S_1^E = -n_m n_1 f_1 \int_{(0)} v(1 - \cos \vartheta) d\Omega = -n_m \nu m f_1.$$  

(5.28)

where the collision frequency $\nu m$ is found through the transport cross section $\sigma' = \int \sigma(1 - \cos \vartheta) d\Omega$ in accordance with Eq. 2.52. In this approximation the integral $S_1^E$ vanishes. The expression 5.28 yields the change in the directed distribution function component $f_1$ as a result of collisions. Its meaning is easy to understand, bearing in mind that $f_1$ characterizes the fraction of electrons moving in the isolated direction (in the $Oz$ direction). Since collisions lead to a substantial change in the direction of electron motion, the characteristic time of variation in $f_1$ must be of the order of the time between collisions $T = 1/\nu$ and its variation rate $\delta f_1/\delta t$ must depend on the product of $f_1$ by the collision frequency $\nu$. This conclusion agrees with Eq. 5.28.

In calculating the integral $S_0$, which determines the change in the isotropic distribution function component, one must take into account the change in the value of electron velocity in collisions. This change is given by the relations obtained in Chapter 2. In elastic collisions of an electron with an atom whose kinetic energy is much less than that of the electron ($K_e \ll K$, the change is equal to

$$v - v' = \left(-\frac{m_e}{m} \right) v(1 - \cos \vartheta)$$

(5.29)

(in distinction to Eq. 2.21, here $v'$ is the electron velocity before collision, and $v$, after it). Since $|v - v'| \ll v$, the integrand of Eq. 5.27 can be represented as

$$v^4 \sigma' f_0 - v^4 \sigma f_0 = (v' - v) \frac{\partial}{\partial v} (v^' \sigma f_0)$$

$$= \frac{m_e}{m} (1 - \cos \vartheta) v \frac{\partial}{\partial v} (v^' \sigma f_0)$$

(5.30)

Substituting Eq. 5.30 into Eq. 5.27, we obtain

$$S_0 = \frac{n_m m_e}{v^2} \frac{\partial}{\partial v} \left[ v^' \int_{(0)} \sigma(1 - \cos \vartheta) d\Omega \right]$$

$$= \frac{n_m m_e}{v^2} \frac{\partial}{\partial v} (v^' \nu m f_0)$$

(5.31)

This expression defines the change in the isotropic distribution function component owing to the velocity reduction (because of energy losses) in elastic conditions.

When the average electron energy is comparable with that of the heavy particles the use of the approximate collision integral 5.25 is illegitimate, and to determine $S_0$ one must substitute the overall expression for the collision integral (Eq. 3.23) into Eq. 5.17. The integral formula obtained can be simplified, inasmuch as the electron velocity greatly exceeds that of the heavy particles and the electron velocity change in collisions is small. Calculations yield the following expression

$$S_0 = \frac{n_m m_e}{v^2} \frac{\partial}{\partial v} (v^' \nu m f_0)$$

(5.32)
for $S_0$:

$$S_0 = \frac{n}{v} \frac{\partial}{\partial v} \left[ \frac{m_e}{m_a} v' \rho(v) + \frac{T_e}{m_a} v' \rho(v) \frac{\partial \rho(v)}{\partial v} \right]$$ (5.32)

where $T_e = \frac{2}{3}(K_e)$ is the temperature of the atoms, which characterizes their average energy. When it is much below the electron energy $T_e \ll m_e v^2$, the second term is also much less than the first, and Eq. 5.32 becomes Eq. 5.31.

The expression 5.32 can be represented as the divergence of a spherically symmetric flux in velocity space. In accordance with the well-known expression for divergence in spherical coordinates we write:

$$S_0 = -\text{div} \cdot \Gamma_\nu = -\frac{1}{v^2} \frac{\partial}{\partial v} \left( v^2 \Gamma_\nu \right)$$ (5.33)

where the density of the flux in the direction of increasing velocity $\Gamma_\nu$ is equal to:

$$\Gamma_\nu = -\rho_0 (n \rho_0) - D_0 \frac{\partial (n \rho_0)}{\partial v}$$ (5.34)

Here $\rho_0 = (m/m_a) \rho_{\nu e}$; $D_0 = (T_e/m_a) \rho_{\nu e}$. The first term describes the reduction in velocity (friction) and the second, the diffusion in velocity space. Substitution of Eqs. 5.33 and 5.34 into the equation for the isotropic distribution function component changes it to the Fokker-Planck equation (see Section 3.3).

To establish the physical meaning of the expression 5.34 we estimate the particle flux across the spherical surface element of the velocity space $dG$ associated with the collisions (Fig. 5.2a). The number of particles in unit volume of configuration space crossing this element at a given velocity increment $\Delta v$ can be found from the equality:

$$dQ = n_f \left( v - \frac{1}{2} \Delta v \right) \Delta v dG = n_f (v) \Delta v dG - \frac{1}{2} n \frac{\partial f_0}{\partial v} (\Delta v)^2 dG$$

where we use the average value of $f_0$ within a volume of height $\Delta v$ and take into account the smallness of the velocity increment on elastic collisions. By correlating this number with the mean time between the collisions $\tau = 1/v$ and by averaging over the collisions we obtain an approximate expression for the flux density*:

$$\Gamma_\nu = \frac{1}{\tau} \frac{dQ}{dG} = n_f (\Delta v) - \frac{1}{2} n \frac{\partial f_0}{\partial v} (\Delta v)^2 \frac{\partial f_0}{\partial v}$$ (5.35)

In accordance with this expression the friction coefficient is proportional to the average velocity decrease,

$$g_\nu = -\nu (\Delta v)$$ (5.36)

and the diffusion coefficient to the mean-square increment,

$$D_\nu = \nu (\Delta v)^2$$ (5.37)

If the change in velocity is random, then $\Delta v = 0$, and Eq. 5.35 retains only the second term determining the diffusion flux. It is directed toward decreasing $f_0(v)$, since with equal probability of positive and negative changes in velocity the flux from the velocity region containing more particles is stronger (see Fig. 5.2b). Therefore diffusion tends to "straighten" the distribution function $f_0$. The whole picture is quite similar to that of diffusion in ordinary configuration space, which results from averaging the random motion of the particles (see Section 7.3).

Let us estimate the coefficients $g_\nu$ and $D_\nu$ for elastic electron-atom collisions. The change in electron velocity on such collisions can be obtained by using the general equation 2.18. Considering that $m_e \ll m_a$ and $v_e \gg v$, we obtain from it:

$$\Delta (v_e^2) = 2v_e \Delta v_e = -2v_0 v_e - \frac{m_e}{m_a} v_e^2 - 2v_e v_s$$

whence

$$\Delta v_e = -\frac{m_e}{m_a} v_e - v_e \cos \varphi$$ (5.38)

where $\varphi$ is the angle between $v_e$ and $v_s$. The first term in Eq. 5.38 depends on the energy losses that are independent of the atom motion.

*For a more rigorous determination of $\Gamma$, one must include the dependence of $\nu$ and $s$ on $v$. 
As seen from Eq. 5.36, the velocity decrease due to these losses \( \Delta v = -(m/m_a)\nu \) results in Eq. 5.34 for the friction coefficient. The second term in Eq. 5.38 is defined by the energy exchange between the electron and atom caused by the motion of the atom. The velocity change on such an exchange is \( \Delta v = v_n \cos \varphi \). With an isotropic atom velocity distribution the average value of this change is \( \langle \Delta v \rangle = 0 \). Therefore the corresponding flux (Eq. 5.35) is associated with a mean-square displacement and represents diffusion in velocity space. Substituting \( \Delta v = v_n \cos \varphi \) into Eq. 5.37 and averaging over \( \varphi \) and \( v_n \) we obtain the expression for the diffusion coefficient:

\[
D = \frac{1}{2} v T_e \langle \cos^2 \varphi \rangle = \frac{1}{2} \frac{v T_e}{m_a}
\]

which differs from Eq. 5.34 only by the coefficient \( \frac{1}{2} \) (the correct numerical coefficient is obtained by a more accurate averaging over the collisions).

Let us now consider the collision terms \( S_0 \) and \( S_1 \) due to inelastic collisions. The electron distribution function may be affected by various inelastic collisions—excitation of different levels, ionization, and recombination. We find here the collision terms for excitation processes in the two extreme cases: when the electron energy greatly exceeds the excitation energy and when these energies are similar. These conditions are often observed in a stationary-gas-discharge plasma, when the average electron energy greatly exceeds the excitation energy of the vibration and rotation levels of the molecule, and is much less than the excitation energy of the electron (atom) levels.

Electron energy losses in inelastic collisions accompanied by excitation are equal to the excitation energy with an accuracy to the small mass ratio \( m_e/m_a \) (see Section 2.1):

\[
\frac{mv^2}{2} - \frac{m^*v^2}{2} = \varepsilon_j
\]

Therefore in the first case, at an excitation energy much less than the electron energy, the velocity change is small:

\[
v - v' = \frac{\varepsilon_j}{m_e v} \ll v
\]

and the collision terms can be obtained in the same way as for elastic collisions. One should use the expression for the collision integral in the form 3.30, taking into account the relation between the volumes of velocity space before and after the collisions, which follows from Eq. 5.39, namely \( v' \, dv' = v \, dv \). Then we obtain, with the aid of Eqs. 5.17 and 5.29,

\[
S_{0a}^{(0)} = -n f \nu_0^{(0)}
\]

where \( \nu_0 \) is the transport frequency of collisions accompanied by excitation of the \( j \) level, and

\[
S_{1a}^{(0)} = \frac{1}{2} \frac{n e}{v} \left( \frac{m_e}{m_a} \nu_0 v_f^{(0)} f_0 \right) = \frac{1}{2} \frac{n e}{v} \left( \frac{m_e}{m_a} \nu_0 v_f^{(0)} f_0 \right)
\]

where \( \nu_0 \) is the averaged energy transfer coefficient, which determines the fraction of energy lost by the electron in inelastic collision.

The sum of the collision terms \( S_{0a}^{(0)} \) and \( S_{1a}^{(0)} \) attributable to inelastic collisions with a low energy loss can be written in a form similar to Eqs. 5.41 and 5.42:

\[
S_{1a}^{(0)} = -n f \nu_0^{(0)}
\]

\[
S_{1a}^{(0)} = \frac{1}{2} \frac{n e}{v} \left( \frac{m_e}{m_a} \nu_0 v_f^{(0)} f_0 \right)
\]

Here \( \nu_0 \) is the summary frequency of inelastic collisions with a low energy loss, and \( \nu_0 \) is the averaged energy transfer coefficient:

\[
\nu_0 = \sum_j \nu_0^{(0)} \quad \kappa_0 = \sum_j \frac{\nu_0^{(0)} \sigma_0^{(j)}}{\sum_j \nu_0^{(0)}}
\]

The summary collision frequency \( \nu_0 \) is usually much less than the frequency of elastic collisions \( \nu_0 \), but the energy transfer coefficient \( \kappa_0 \) in molecular gases may be much larger, \( \kappa_0 = 2m_e/m_a \).

In the second case, when the electron energy exceeds the excitation energy only slightly, one can assume that an inelastic collision results in a total loss of electron energy. With this assumption the collision integral retains only the term defining particle removal from the preassigned velocity range. It takes the form

\[
S_{0a}^{(0)} = -\int_{\Delta v} n f v_0 \sigma_0 \, d\Omega = -n f \nu_0^{(0)}
\]

where \( \nu_0 = n v \int \sigma_0 \, d\Omega \) is the frequency of inelastic collisions of the given type. The arrival of particles in the low-energy region can be covered by adding the term

\[
S_{0a}^{(0)} = \frac{\delta(v)}{4\pi v^2} Q^{(0)}
\]
where

\[ Q^{(i)} = - \int_{v}^{(i)} \left( \frac{\delta n_f}{\delta t} \right) d^3v = n_e \int_{v}^{(i)} f^{(i)} d^3v \]

is the total number of inelastic collisions of the given type, and \( \delta(v) \) is the delta function, which differs from zero only when \( v = 0 \). The integral over the velocity space volume from Eq. 5.46 is equal to

\[ \int_{v}^{(i)} f^{(i)} d^3v = 4\pi \int_{0}^{\infty} f^{(i)} v^2 dv = Q^{(i)} \]

Substituting the collision term 5.45 into Eqs. 5.17 and 5.19, we find

\[ S_{v}^{(i)} = -n f^{(i)} v^{(i)}; \quad S_{v}^{(i)} = -n f^{(i)} v^{(i)} \] (5.47)

Similar equations are obviously obtained for collision terms describing ionization at an electron energy close to that of ionization. Then the increased number of electrons can be included by doubling the term 5.46, which reflects the appearance of particles in the low-velocity region. Summing Eq. 5.47 over all the inelastic collisions with a large energy loss results in replacement of the frequency \( v^{(i)} \) by the summary frequency of inelastic collisions:

\[ S_{v}^{(i)} = -n \nu^{(i)} f; \quad S_{v}^{(i)} = -n \nu^{(i)} f \] (5.48)

where \( \nu^{(i)} = \Sigma \nu^{(i)} \). In many cases \( \nu^{(i)} < \nu^{(e)} \), and the contribution of inelastic collisions to the collision term \( S_{v}^{(i)} \) is insignificant \( (|S_{v}^{(i)}| \ll |S_{v}^{(i)}|) \). At the same time they may determine, almost completely, the isotropic collision term \( S_{v}^{(i)} = \nu^{(i)} f \) in accordance with Eq. 5.31.

The expressions obtained define the collision terms related to electron-atom collisions. The terms determined by electron-ion collisions \( S_{v}^{(i)} \) and \( S_{v}^{(i)} \) have the same form. The expressions for the collision terms due to electron-electron collisions can be represented only in integral form. We do not give them here, but note that the equation for \( S_{v}^{(i)} \) is derived from the general elastic collision integral 3.23 by replacing the total distribution function \( f(v) \) with the isotropic component \( f(v) \).

5.4 DISTRIBUTION FUNCTION OF ELECTRONS IN ELECTRIC FIELD WITH DETERMINING EFFECT OF ELASTIC ELECTRON-ATOM COLLISIONS

Under stationary conditions (when the distribution function is time invariant) the equations for the components of the distribution functions in a constant electric field (Eqs. 5.16 and 5.18) take the form

\[ -\frac{e E}{3m_e v^2} \frac{d}{dv} (v^2 f) = S_{v}^{(i)}; \quad -\frac{e E}{m_e} \frac{d}{dv} f = S_{i} \] (5.49)

Equation 5.53 describes the stationary balance between the acquisition of a directed velocity in an electric field and its losses on collisions. It

The collision terms \( S \) (Eq. 5.24) constitute the sum of the terms for collisions of electrons with atoms, ions, and each other, \( S = S_{v}^{(i)} + S_{v}^{(i)} + S_{v}^{(i)} \).

To estimate the components \( S_{v}^{(i)} \) and \( S_{v}^{(i)} \) we can use the order equalities

\[ S_{v}^{(i)} \approx \nu f; \quad S_{v}^{(i)} \approx \kappa e f \] (5.50)

These follow from the determination of the quantity \( S_{v}^{(i)} = \delta n f / \delta t, S_{v}^{(i)} = \delta n f / \delta t \). The former characterizes the change of electron velocity direction in collision and therefore depends on the collision frequency \( \nu \), and the latter characterizes the change in velocity and is determined by the product of the collision frequency by the energy transfer coefficient \( \kappa e \). For electron-heavy particle collisions the relations 5.50 are obtained directly from the equations of Section 5.3. With their aid one can evaluate the relative role of the different types of collision.

Let us consider the solution of Eq. 5.49 for a plasma with a low degree of ionization, when only electron-atom collisions are substantial. The conditions under which the effect of electron-electron and electron-ion collisions on the distribution function can be neglected are found from Eq. 5.50:

\[ \nu \approx \nu^{(e)} \ll \nu^{(v)}; \quad \nu \approx \kappa e \nu \ll \nu^{(v)} \] (5.51)

(It is taken into account here that \( \kappa e \approx \kappa \) since \( m_e = m_e \) and \( \kappa \approx 1 \).) The second inequality is more rigid than the first. For elastic collisions it takes the form \( \nu \ll \nu^{(v)} \) or

\[ \eta = \frac{n_e}{n_a} \ll \frac{m_e}{m_a} \frac{\nu^{(v)}}{\nu^{(v)}} \] (5.52)

It can be seen that only at very small values of \( \eta \) \( (\eta \ll m_e/m_a) \) is it possible to neglect the effect of electron-electron collisions on the distribution function.

We first find the solution of Eq. 5.49 for the case where only elastic electron-atom collisions are substantial. Substituting Eqs. 5.28 and 5.32 for \( S_{v}^{(i)} \) and \( S_{v}^{(i)} \) into Eqs. 5.49, we obtain

\[ \frac{e E}{m_e} \frac{d}{dv} f = \nu f; \quad \frac{e E}{3m_e v^2} \frac{d}{dv} (v^2 f) = \frac{1}{2} \frac{n}{v^2} \frac{d}{dv} \left[ \kappa e v^2 \left( v f + \frac{T_e}{m_e} f \right) \right] \] (5.54)

Equation 5.53 describes the stationary balance between the acquisition of a directed velocity in an electric field and its losses on collisions. It
enables one to find the relationship between the functions $f_1$ and $f_0$:

$$f_1 = \frac{eE}{m_e v'} \frac{df_0}{dv}$$  \hspace{1cm} (5.55)

Using Eq. 5.55, we obtain from Eq. 5.13 the general expression for the directed electron velocity:

$$u_E = \frac{4\pi}{3} \int_0^\infty f_1(v) v^3 dv = \frac{4\pi eE}{e m_e} \int_0^\infty \frac{v^3 df_0}{v^2 dv}$$

$$= -\frac{4\pi eE}{3 m_e} \int_0^\infty f_0(v) \frac{d}{dv} \left( \frac{v}{\tau} \right) dv$$  \hspace{1cm} (5.56)

(the last equality results from integration by parts). If the collision frequency is velocity independent, that is, if $\nu = \text{const}$, expression 5.56 leads to an equation independent of the form of $f_0(v)$:

$$u = -\frac{eE}{m_e v'} 4\pi \int_0^\infty f_0(v) v^3 dv = -\frac{eE}{m_e v'}$$  \hspace{1cm} (5.57)

The equation for the isotropic distribution function component is found by substituting Eq. 5.55 into Eq. 5.54:

$$2 \frac{e^2 E^2}{3 m_e v'^2} \frac{df_0}{dv} = -D_0 \frac{d}{dv} \left( \frac{r}{m_e} \frac{df_0}{dv} \right)$$  \hspace{1cm} (5.58)

The left-hand side of the equation, which defines the electron velocity increase under the effect of the electric field, can, as well as the right-hand ("collision") side, be represented as a divergence of a spherically symmetric flux. The flux

$$\Gamma_{\nu} = -\frac{2}{3} \frac{e^2 E^2}{m_e v'^2} \frac{df_0}{dv} = -D_0 \frac{d}{dv} \left( \frac{r}{m_e} \frac{df_0}{dv} \right)$$  \hspace{1cm} (5.59)

is proportional to the derivative $df_0/dv$, and since $f_0(v)$ is a decreasing function the flux is directed toward the higher velocities. The expression 5.59 describes the diffusion in velocity space. The energy accumulation by electrons in the electric field is of a diffusion nature because it continues for many periods between collisions. As a result, the acceleration during each period is added vectorially with an arbitrarily directed random velocity, and hence the change in velocity is accidental. It is easy to estimate directly the diffusion coefficient characterizing this process. The change in electron velocity vector within the intercollisional time $\tau$ is due to the electron acceleration in the electric field:

$$\Delta v_E = v' - v = -\frac{eE}{m_e} \tau = -\frac{eE}{m_e}$$

The corresponding velocity change $\Delta v$ is given by the relations

$$v'^2 = v^2 + 2v \Delta v_E; \hspace{1cm} \Delta v = v' - v = \frac{\Delta v_E}{v}$$

which take into account that $\Delta v_E \ll v$. Since the velocity $v$ can be directed arbitrarily, the average value of $\Delta v$ reduces to zero. The average value of $(\Delta v)^2$ is equal to

$$\langle (\Delta v)^2 \rangle = \frac{1}{3} \langle (\Delta v_E)^2 \rangle = \frac{1}{3} \frac{e^2 E^2}{m_e v'}$$

whence we find the diffusion coefficient (see Eq. 5.37):

$$D_v = \langle (\Delta v)^2 \rangle = \frac{1}{2\pi} \frac{e^2 E^2}{6 m_e v'}$$  \hspace{1cm} (5.60)

which agrees with Eq. 5.59 to within a numerical factor.

Thus it follows from Eq. 5.58 that the divergence of the density of the summary flux in velocity space—the flux determining energy acquisition in the field $\Gamma_v$ (Eq. 5.59) and the flux determining the energy losses on collisions $\Gamma_{\nu}$ (Eq. 5.34)—reduces to zero; that is,

$$\frac{1}{v'} \frac{d}{dv} \left[ v' (\Gamma_v + \Gamma_{\nu}) \right] = 0$$  \hspace{1cm} (5.61)

Integrating the equation, we arrive at the constancy of the summary flux across the spherical surface of velocity space $4\pi v'^2 (\Gamma_v + \Gamma_{\nu}) = C$

Since the quantities $\Gamma_v$ and $\Gamma_{\nu}$ are finite as $v \rightarrow 0$, the integration constant $C$ is zero. This means that the summary flux is zero as well. By using the expressions for its components $\Gamma_{\nu}$ (Eq. 5.59) and $\Gamma_v$ (Eq. 5.34), we obtain the equation for $f_0$, which stems from Eq. 5.58:

$$2 \frac{e^2 E^2}{3 m_e v'^2} \frac{df_0}{dv} + \kappa_v \left( u_0 + \frac{T_e}{m_e} \frac{df_0}{dv} \right) = 0$$  \hspace{1cm} (5.62)

The solution of Eq. 5.62 has the form

$$f_0 = A \exp \left[ -\int_0^v \frac{m_e v dv}{T_v + 2eE^2/3\kappa_v (v')^2 m_e} \right]$$  \hspace{1cm} (5.63)

where the constant $A$ is found from the normalization condition (Eq. 5.12):

$$4\pi \int_0^\infty f_0 v^2 dv = 1$$

Equation 5.63 indicates the nature of the deviations of the electron velocity distribution from equilibrium. It can be seen that these devia-
The directed velocity can be found by substituting Eq. 5.67 into Eq. 5.56. Integrating, we obtain

\[ u = 0.69 \kappa \frac{1}{\sqrt{\kappa}} \left( \frac{e E}{m_e} \right) \]  

(5.69)
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Fig. 5.4 Changes in distribution functions due to inelastic collisions.

electrons lose an energy approximately equal to $\delta_1$ and move from the region of above-threshold energies to the low-energy region. Figure 5.4 shows a curve characterizing the reduction rate of distribution function as a result of this process $\delta f = -\nu^0 f_0 \delta t$ in the high-energy region, and the reverse curve $\delta f^* = -\delta f$ defining the arrival of electrons in the low-velocity region. The width of these curves depends on the rate of decrease in $f_0$ in the inelastic region ($K_e > \delta_1$). At an excitation energy greatly exceeding the average electron energy, $\delta f$ falls off very fast, and the curve $\delta f^*$ becomes narrow, "pressed" against the excitation threshold; accordingly, the curve $\delta f^*$ is "pressed" against the origin.

To find the change in function $f_0$ due to inelastic collisions, one must take into account not only transitions of $\delta f^*$ and $\delta f$, but also the diffusion in velocity space due to elastic collisions. The diffusion smooths the curve $f_0(v)$. In particular, it increases the gradient of $f$ in the region of $v < \nu_1$ and partly replenishes the loss of electrons from the high-velocity region (see Fig. 5.4). In actual conditions several inelastic processes with different excitation energies are usually substantial. The curves $\delta f$ associated with such processes are added together, and the drop of the function in the inelastic collision region is increased still further.

Let us consider quantitatively the effect of inelastic collisions on the distribution function. We begin with finding the distribution function in the region of velocities exceeding the inelastic-process threshold $v > \nu_1 = (2E_e/m_e)^{1/2}$, with the aid of Eqs. 5.49:

$$-(\frac{eE}{3m_e v^2}) \frac{df_0}{dv} = S_0; \quad -(\frac{eE}{m_e}) \frac{df_0}{dv} = S_1 \quad (5.70)$$

As noted before, when the distribution function falls off rapidly inelastic collisions of each type are experienced mainly by electrons whose energy is close to the excitation energy of the corresponding levels. Therefore we can use the expression for inelastic collision terms (Eq. 5.47) obtained by assuming the total electron energy loss. The overall equation for $S_0$ and $S_1$ then takes the form

$$S_0 = \frac{nm_e \nu_1^0 \nu^2 f_0}{m_e v^2} \left( \nu_{v_e}^0 v_{v_e}^2 f_0 - \nu_{v_e} f_0 \right);$$

$$S_1 = -\left( \nu_{v_e} + \nu_{v_e}^0 \right) n f_1 \quad (5.71)$$

where $\nu_{v_e}^0$ is the summary frequency of inelastic collisions with a large energy loss (Eq. 5.48).

In the inelastic collisions region the summary frequency usually lies largely within the following limits (see Chapter 2):

$$\left( \frac{m_e}{m_e} \nu_{v_e} \right) \nu_{v_e}^0 \ll \nu_{v_e} \ll \nu_{v_e}^0 \quad (5.72)$$

and owing to the smallness of $m_e/m_e$, the inequality 5.72 is valid beginning with velocities close to the threshold of inelastic processes. With this in mind we simplify Eqs. 5.71. In the equation for $S_0$ we neglect the term representing elastic scattering (it is of the order of $\nu^0 m_e/m_e n f_0$), and in the equation for $S_1$, the term for inelastic collisions. Then

$$S_0 = -\nu_{v_e}^0 n f_0; \quad S_1 = -\nu_{v_e} n f_1 \quad (5.73)$$

Substituting $S_1$ into the equation for $f_1$ yields the former relationship 5.55 between $f_1$ and $f_0$. Using it, we obtain from Eq. 5.70 the equation for $f_0$ in the inelastic collision region:

$$\frac{e^2 E^2}{3m_e v^2} \int_0^{\nu_{v_e}} \left( \frac{v^2 f_0}{v_{v_e}^0} \right) \frac{df_0}{dv} - \nu_{v_e} f_0 = 0 \quad (5.74)$$

It can be written differently:

$$\frac{df_0}{dv} + \frac{2}{v_{v_e}} \frac{1}{\nu_{v_e}} \frac{df_0}{dv} - \frac{q^2}{v_{v_e}^2} f_0 = 0 \quad (5.75)$$

where

$$q^2 = \frac{3m_e v_{v_e}^0}{e^2 E^2} \nu_{v_e} \nu_{v_e}^0 v_{v_e}^0 \quad (5.76)$$

The quantity $q$ determines the decrease of the distribution function at $v > \nu_1$. We assume

$$q^2 = \frac{m_e v_{v_e}^0}{m_e v_{v_e}^0} \frac{v_{v_e}^2}{e^2 E^2} \nu_{v_e} \nu_{v_e}^0 \frac{E}{m_e} > 1 \quad (5.76)$$

where $T_e = e^2/2m_e(e^2/2m_e)^{1/2}$ is the average electron energy in an electric
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field in the absence of inelastic collisions. It is seen that this inequality follows from Eq. 5.72 at $T_0 \ll \bar{E}$, In solving Eq. 5.75 we can use, with the aid of Eq. 5.76, an approximation based on the fact that $f_0$ is a rapidly decreasing function of $v$. Therefore, in Eq. 5.75 the second term can be neglected compared with the first $[(1/v) df_0/dv] < (df/dv)^2$, and $q$ can be assumed a slowly varying function of the velocity $(1/q) dq/dv$. Then we obtain an approximate solution of the equation, which reduces to zero as $v \to 00$, in the form

$$f_0 = C \exp \left[ - \int_{v_1}^{v} \frac{q}{v} dv \right] = C \exp \left[ - \frac{\sqrt{3}m_e}{eE} \int_{v_1}^{v} \sqrt{v^2 - v_i^2} dv \right] \quad (5.77)$$

This solution holds in the velocity region bounded by the inequality $v^h \approx m_e v^i/m_e$, when the electron energy losses are due to inelastic collisions.

When the opposite inequality is true, one can use $f_0$ obtained without allowance for inelastic collisions (Eq. 5.64). The coefficient $C$ is determined approximately by combining Eq. 5.77 with Eq. 5.64 at the region boundary for $v' = m_e v'/m_e$. This boundary is usually very close to the threshold of inelastic processes, and the combination may be done at the threshold velocity $v = v_1$.

The particular form of $f_0$ (Eq. 5.77) is determined by the velocity dependence of the collision frequencies $v^i$ and $v^h$. Let us find it for $v^i = \text{const}$.

The dependence $v^h(v)$ near the threshold of inelastic processes can be approximated by the equation

$$v^h(v) = n_0 S^h v_1 \approx v_1 \frac{v}{v_1 - 1} \quad (5.78)$$

Using this approximation, we can compute the power index in Eq. 5.77:

$$\frac{\sqrt{3}m_e}{eE} \int_{v_1}^{v} \sqrt{v^2 - v_i^2} dv = \frac{\sqrt{3}m_e v_1}{eE} \int_{v_1}^{v} \left( \frac{v}{v_1 - 1} - 1 \right) dv = q_0 \left( \frac{v}{v_1} - 1 \right)^{3/2}$$

where

$$q_0 = \frac{2}{3} \frac{m_e v_1^2}{eE} \quad (5.79)$$

Substituting Eq. 5.79 into Eq. 5.77, we obtain the expression for $f_0$ at $v > v_1$:

$$f_0 = C \exp \left[ - q_0 \left( \frac{v}{v_1} - 1 \right)^{3/2} \right] \quad (5.80)$$

This approximation is sometimes called quasi-classical by analogy with the quasi-classical approximation in quantum mechanics.

Using Eq. 5.80, we can find the efficiency of the processes determined by the electrons whose energy exceeds the inelastic-collision threshold. We first calculate the average number of inelastic collisions per unit time (see Eq. 5.46):

$$Q_n = n_e \int f_0^h d^3 v = 4\pi n_e \int v^h f_0^h(v) v^2 dv$$

Substituting the approximation 5.78 and the function 5.80 and taking into account the fast decrease of $f_0$ and $v > v_1$, we find

$$Q_n = 4\pi C n_e v_0^h \int v^2 \left( \frac{v}{v_1} - 1 \right) \exp \left[ - q_0 \left( \frac{v}{v_1} - 1 \right)^{3/2} \right] dv \approx 4\pi C n_e v_0^h v_1^4 \int \left( \frac{v}{v_1} - 1 \right) \exp \left[ - q_0 \left( \frac{v}{v_1} - 1 \right)^{3/2} \right] dv \quad (5.81)$$

and further

$$Q_n \approx 4\pi C n_e v_0^h q_0^{4/3} v_1 \int x \exp(-x^{3/2}) \, dx \approx 2.5\pi C n_e v_0^h q_0^{4/3}$$

Let us now find the average ionization frequency characterizing the rate of appearance of new electrons in the plasma. To this end we use the approximation of the frequency of collisions leading to ionization (see Eq. 2.86):

$$\nu^i = n_0 s^i v = v_1 \frac{v^2 - v_i^2}{v_i^2} \quad (5.82)$$

where $v_i = (2e\bar{E}/m_e)^{1/2}$ is the threshold velocity for ionization. The average ionization frequency is obtained by averaging Eq. 5.82 with the aid of the function $f_0$ (see Eq. 5.80). Using Eq. 5.11, we obtain

$$\nu^i = 4\pi \int_{v_1}^{v} \nu^i(v) f_0(v) v^2 dv \quad (5.83)$$

$$= 4\pi C v_1^4 \int_{v_1}^{v} \exp \left[ - q_0 \left( \frac{v}{v_1} - 1 \right)^{3/2} \right] \left( \frac{v^2}{v_i^2} - 1 \right) v^2 dv$$

When calculating the integral of Eq. 5.83 we take into account that $f_0(v)$ falls off rapidly, and the integrand function $\nu^i f_0$ approaches the ionization threshold. Therefore it can be assumed that under the integral $v - v_1 = \omega \ll v_1$, and then the exponent contained in it has the form

$$\exp \left[ - q_0 \left( \frac{v}{v_1} - 1 \right)^{3/2} \right] \approx \exp \left[ - q_0 \left( \frac{v}{v_1} - 1 \right)^{3/2} \right] \exp(-\xi \omega)$$

where $\xi = \frac{3}{2} \left( q_0/v_1 \right) (v/v_1 - 1)^{3/2}$. Assuming also that under the integral...
where \( T_0 \) and \( v_0 \) are the temperature determining the average electron energy in the field \( E \) without inelastic collisions and \( A = 3Qm_e^2v_0^2/4\pi ne^2E^2 \approx 1.4C_0q_0^2 \). For \( v = v_0 \) we must combine this solution with Eq. 5.77. Since \( f(v) \to 0 \), we find

\[
B \approx C/A \exp \left( \frac{\tilde{E}_1}{T_0} \right) \approx 0.71 \frac{v_0}{q_0^2} \exp \left( \frac{\tilde{E}_1}{T_0} \right) \tag{5.88}
\]

From Eq. 5.87 it follows that for \( v \to 0 \) the value of \( f_0 \) tends to infinity as \( 1/v \). This conclusion is obviously associated with the assumption of a total loss of energy by the electrons on inelastic collisions, as a result of which we introduced an electron source \( Q_0 \delta(v) \), infinite with respect to the amplitude, for \( v = 0 \). Note that although \( f_0 \) has a singularity as \( v \to 0 \), in this case the distribution function for total velocities \( f_s = 4\pi v^2f_0 \) remains small, since \( f_s - v^2, f_0 \sim v \); therefore this singularity does not cause any difficulty in averaging.

Let us first consider the case \( T_0 \to \tilde{E}_1 \). When analyzing Eq. 5.87 we can clearly see that the integral appearing in it depends on the lower limit only in the low-velocity range \( v < v_1 \). For these ranges we can use the approximate normalization condition, which allows for the smallness of the deviation of \( f_0(v) \) from Eq. 5.64 at \( v < v_1 \) and its fast drop at \( v > v_1 \). Therefore the normalized coefficients can be found approximately from the condition

\[
4\pi_0 \int f_0(v) v^2 \, dv \to 4\pi_0 \int f_0(v) v^2 \, dv = 1
\]
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according to which \( D = (m_e/2\pi T_0)^{3/2} \).

\[
C = f_0(v) = \frac{m_e}{2\pi T_0} \exp \left( \frac{E_0}{T_0} \right) \left( 1 + 0.70 \frac{T_0}{E_0} q_0^{3/2} \right)^{-1}
\]

(5.90)

Thus at \( T_0 \ll \mathcal{E}_1 \), in most of the region \( v < v_1 \), the distribution function is found to be the same as with elastic collisions alone (for \( v' = \) const, it is Maxwellian). The deviations are substantial only at low velocities \( [v \ll v_1(\mathcal{E}_1/T_0) \exp(-\mathcal{E}_1/T_0)] \) and at velocities close to \( v_1 \) \((v_1 - v < v(T_0/\mathcal{E}_1)) \).

We have determined the function \( f_0(v) \) for \( T_0 \ll \mathcal{E}_1 \). An increase in electric field strength and \( T_0 \) increases the number of electrons with energies exceeding the inelastic threshold. The effect of inelastic collisions on the form of the distribution function at \( v < v_1 \) increases correspondingly. Let us consider the case

\[
T_0 = \frac{2}{3} \frac{e^2 E^2}{m_e \mathcal{E}_1(v/v_1)^{3/2}} \gg \mathcal{E}_1
\]

assuming, however, that \( T_0 \ll \mathcal{E}_1(v_1/v)^4 \). Then the parameter \( q_0 \) of Eq. 5.79 remains large, and the solution of Eq. 5.80 for the inelastic-collision range \((v > v_1)\) holds true. In the range of \( v < v_1 \) the equation for the distribution function (Eq. 5.86) is simplified because the second term, which describes the energy losses on elastic collisions, is negligibly small as compared with the first one. The mechanism of energy distribution formation is as follows: at \( v < v_1 \) the electrons accumulate energy as a result of velocity diffusion in the electric field (see p. 124) and lose it at \( v > v_1 \) owing to inelastic collisions. For \( \mathcal{E}_1 < T_0 \) we can assume the exponents in Eq. 5.89 to be equal to unity. The function \( f_0 \) then acquires the simple form

\[
f_0 = A \left[ \frac{1}{v} - \frac{1}{v_1} (1 - 0.71 q_0^{-2/3}) \right]
\]

(5.91)

The coefficient \( A \) can be found from the approximate normalization condition:

\[
1 = 4\pi \int_0^{v_1} f_0(v^2) dv = \frac{2}{3} \pi A v_1^2 (1 - 1.42 q_0^{2/3})
\]

(5.92)

For the distribution obtained the average electron energy is equal approximately to

\[
\langle E_e \rangle \approx \frac{m_e v_1^2}{2} - 4\pi \int_0^{v_1} f_0(v) dv \approx \frac{3}{10} \mathcal{E}_1
\]

(5.93)

*Note that both conditions can be correlated for a heavy-atom gas, since the coefficient \( \kappa_{ee} = 2m_e/m_a \) is very small.

(we made an allowance for the fact that \( f_0(v) \) drops fast at \( v > v_1 \) and \( q_0 \gg 1 \)). The distribution function at the boundary of the inelastic region is given by the relation

\[
C = f_0(v) = \frac{0.34}{v_1 q_0^{3/2}}
\]

Using this, we can find from Eq. 5.11 the average ionization frequency for the case under consideration:

\[
v'_{\nu_0^B} = \frac{3.8}{q_0^{3/2}} \left( 1 - \frac{E_0}{\mathcal{E}_1} \right)^{-1} \exp \left[ -q_0 \left( \frac{E_0}{\mathcal{E}_1} - 1 \right)^{3/2} \right]
\]

(5.94)

where \( q_0 = (2/\sqrt{3})(m_e \sqrt{\nu_0^B E}/e) \).

We have considered the effect on the distribution function of inelastic collisions resulting in the excitation of different levels and the loss of electron energy. In principle, reverse processes are also possible, namely, electron-excited atom collisions, which increase the electron energy (second-type collisions). One can frequently assume that they are insignificant, since the excited atoms lose their energy mainly through radiation and collisions with neutral particles and the walls. Sometimes, however, such as at high neutral gas pressures, when the plasma is not completely transparent to radiation, one has to take into account second-type collisions. They lead to some compensation for the energy losses caused by excitation. In the limiting case, on changeover to a closed system, forward and reverse inelastic collisions ensure such energy exchange between electrons and atoms, which leads to equilibrium Maxwellian distribution (see Section 4.1).

5.6 EFFECT OF ELECTRON-ELECTRON COLLISIONS ON ELECTRON DISTRIBUTION FUNCTION

So far we have neglected the effect of electron-electron collisions on the distribution function. As shown in Section 5.4, the conditions under which this neglect (see Eq. 5.52) is admissible are fulfilled only for very low degrees of ionization. Let us now consider the opposite, when energy exchange of electrons as a result of electron-electron collisions is much more efficient than the energy losses in electron-atom collisions. This occurs when

\[
\kappa_{ee} \nu_ee \gg \nu_{ee}
\]

(5.95)

If the electron energy losses are caused by elastic collisions and \( \kappa_{ee} = 2m_e/m_a \), the condition 5.95 takes a form opposite to Eq. 5.52:

\[
\eta > \frac{m_e \delta_{ee}}{m_a \delta_{ee}}
\]

(5.96)
It can be seen that this inequality holds true even at comparatively low degrees of ionization, which are lower, the smaller the electron energy.

Bearing in mind the condition 5.95, let us discuss the solution of the equation for the isotropic distribution function component. In accordance with Eq. 5.49 this equation has the form

\[
\frac{eE_n}{3m_e \nu^2} \frac{d}{d\nu} \left( \nu^2 f_1 \right) = S_{bo} + S_{ov} + S_{oe}
\]

(5.97)

In the equation, the stationary balance of the average electron energies is determined by the first three terms—the acquisition of energy from the electric field by electrons (described by the first term) is offset by electron energy losses in collisions with atoms and ions (described by the second and third terms). Accordingly, the first term must be of the same order of magnitude as the sum \(S_{bo} + S_{ov}\). The collision term \(S_{oe}\) due to electron-electron collisions clearly cannot affect the variation in the average energy of the electrons; it only results in redistribution of energy among them. In our case the term \(S_{oe}\) is of the order of \(\nu_{me} n_f o\); it greatly exceeds the second (\(S_{bo} = \kappa_{ee} \nu_{me} n_f o\)) and the third (\(S_{ov} = \kappa_{ve} \nu_{me} n_f o\)) terms. Therefore, to a first approximation the form of the distribution functions is described by the equation

\[
S_{bo} = \frac{1}{2} n_e^2 \int_0^\infty \left( f_f (\nu) f_f (\nu) - f_f (\nu) f_f (\nu) \right) d\nu = 0
\]

(5.98)

As noted in Section 3.3, we can write \(S_{bo}\) in the form of the collision integral in Eq. 3.23, replacing \(f_f (\nu)\) by \(f_f (\nu)\). As shown in Section 4.2, this equation leads to the Maxwellian velocity distribution:

\[
f_0 = \left( \frac{m_e}{2 \pi T_e} \right)^{3/2} \exp \left( -\frac{m_e \nu^2}{2 T_e} \right)
\]

(5.99)

The electron temperature \(T_e\), however, cannot be found from Eq. 5.98. Since this equation includes only interelectron collisions, it is satisfied by the distribution 5.99 with any value of \(T_e\).

To find \(T_e\) one can use the energy balance equation, which is derived from the initial equation 5.97 for \(f_0\). In order to obtain it we multiply Eq. 5.97 term by term by \(m_e \nu^2/2\) and by the weighting factor \(4 \pi \nu^2 dv\) and integrate over the velocities from zero to infinity. The integral of the first term is calculated by integrating by parts:

\[
\int_0^\infty \frac{m_e \nu^2}{2} 4 \pi \nu^2 dv \left[ \frac{eE_n}{3m_e \nu^2} \frac{d}{d\nu} \left( \nu^2 f_1 \right) \right] = -\frac{2 \pi}{3} \frac{eE_n}{2} \int_0^\infty \nu^2 \frac{d}{d\nu} \left( \nu^2 f_1 \right) dv
\]

(5.100)

\[
= \frac{neE}{3} \int_0^\infty \nu^2 f_1 dv = neE
\]

(Here we used the general equation 5.21 for \(u\).) The collision term that is determined by elastic electron-atom collisions (Eq. 5.32) with a Maxwellian electron velocity distribution (Eq. 5.99) is equal to

\[
S_{bo} = \frac{1}{2} \frac{\kappa_{ee} \nu_{me} n_f o}{\nu^2} \left[ v v_{me} f_0 + \frac{T_e}{m_e \nu^2} \frac{d f_0}{d\nu} \right]
\]

(5.101)

The integral in which it appears is calculated similarly to Eq. 5.100:

\[
\int_0^\infty S_{bo} \frac{m_e \nu^2}{2} 4 \pi \nu^2 dv = -2 \pi m_e \nu_{me} \left( 1 - \frac{T_e}{T_e} \right) \int_0^\infty \nu^2 v_{me} f_0 dv
\]

(5.102)

\[
= \frac{3}{2} n \kappa_{ee} \nu_{me} f_0 (T_e - T_e)
\]

due allowance is made for the fact that

\[
4 \pi \int_0^\infty \frac{m_e \nu^2}{2} 2 v^2 f_0 dv = \frac{3}{2} T_e
\]

(5.103)

The integral containing the collision term for inelastic collisions with small energy losses is calculated in the same way. Using Eq. 5.43, we find

\[
\int_0^\infty S_{oi} \frac{m_e \nu^2}{2} 4 \pi \nu^2 dv = \frac{3}{2} n \kappa_{ve} \nu_{ve} f_0 T_e
\]

(5.104)

where the bar denotes, as before, averaging over \(v\) with a weight of \(v^2\). The integral due to inelastic collisions with a large energy loss is found by using Eq. 5.48:

\[
\int_0^\infty S_{oi} \frac{m_e \nu^2}{2} 4 \pi \nu^2 dv = -4 \pi m_e \frac{m_e \nu^2}{2} \nu_{me} f_0 \nu_{me}^2 dv
\]

(5.105)

Finally, it is easy to show that the integral containing \(S_{bo}\) characterizes the summary change in electron energy due to electron-electron collisions (see p. 161). Therefore it vanishes irrespective of the type of the
distribution function. Summing up the expressions 5.102–5.105, which are associated with electron–atom collisions, we find

\[
\int_0^\infty \frac{m_v v^2}{2} S_0 d v = \frac{3}{2} n \kappa_{0a} \nu_{0a} (T_e - T_a) - \frac{3}{2} n \kappa_{0a} \nu_{0a} T_e - \frac{3}{2} n \nu_{0a} T_e
\]

\[
= -\frac{3}{2} n \kappa_{0a} \nu_{0a} (T_e - T_a) \tag{5.106}
\]

where we introduce the following notation:

\[
\kappa_{0a} \nu_{0a} = \kappa_{0a} \nu_{0a} + \kappa_{0a} \nu_{0a} + \nu_{0a} \; ; \quad \nu_{0a} = \nu_{0a}^+ + \nu_{0a}^+ + \nu_{0a}^- \tag{5.107}
\]

and take into account that inelastic collisions are usually substantial only at \( T_e \gg T_a \).

A similar expression is obtained for the integral containing the collision term \( S_0 \):

\[
\int_0^\infty S_0 (m_v v^2) d v = -\frac{3}{2} n \kappa_{0a} \nu_{0a} (T_e - T_i) \tag{5.108}
\]

The energy balance equation is obtained by equating Eq. 5.100 to the sum of Eqs. 5.106 and 5.108:

\[
-neuE = \frac{3}{2} n \kappa_{0a} \nu_{0a} (T_e - T_a) + \frac{3}{2} n \kappa_{0a} \nu_{0a} (T_e - T_i) \tag{5.109}
\]

This represents the equality of the energy input per unit volume of the electron gas, and the energy lost by the electrons per unit volume. The input energy in the electric field is given by the conventional formula derived from the Joule–Lentz law \( P_E = j \overline{E} \), where \( j \) is the electron current density equal to \(-neu\). The losses of energy in Eq. 5.109 are due to its transfer from electrons to atoms and ions in collisions.

For the balance equation 5.109 to be complete, one must obtain the expression for the directed velocity \( u \). The relationship of the directed component of the distribution function \( f_\parallel \) with \( f_0 \), is given by the second equation of Eqs. 5.49. For the conditions \( \nu_{0e} = \nu_{0e} = \nu_{0e} \), one can use the previous expression for the collision term \( S_0 \), (Eq. 5.28), and, accordingly, the previous equation 5.55 for \( f_\parallel \). The directed velocity is given by Eq. 5.56, which takes the following form for the Maxwellian distribution (Eq. 5.99):

\[
u = \frac{4 \pi e E}{3 m_e} \int_0^\infty \frac{v^2}{v_{0a}} d v = \frac{4 \pi}{3} \left( \frac{m_e}{2 \pi T_e} \right)^{3/2} e E \int_0^\infty \frac{v^4}{v_{0a}} \exp \left( -\frac{m_e v^2}{2 T_e} \right) d v \tag{5.110}
\]

\[
= \frac{4 \pi}{3} \left( \frac{m_e}{2 \pi T_e} \right)^{3/2} \frac{e E}{T_e} \int_0^\infty \frac{v^4}{v_{0a}} \exp \left( -\frac{m_e v^2}{2 T_e} \right) d v \tag{5.111}
\]

where the coefficient \( \gamma \) is determined by the form of the dependence

\[
\gamma = \frac{1}{\sqrt{6} \kappa_{0a} T_e} \tag{5.112}
\]

At \( v' = \text{const} \), integration yields \( u = eE/m_e v_{0a} \) (see Eq. 5.57). The directed velocity can generally be expressed through averaged collision frequencies (Eq. 5.103) in a similar way:

\[
u = -\frac{\gamma eE}{m_e v_{0a}} \tag{5.113}
\]

where \( \gamma \) is a numerical coefficient of the order of unity. In particular, when the collision frequency is proportional to the velocity \( \nu_{0a} = v/v_{0a} \), we find, with the aid of Eqs. 5.103 and 5.110,

\[
\nu_{0a} = \frac{4}{\sqrt{2 \pi}} \frac{1}{m_e \kappa_{0a}} \sqrt{T_e/m_e} \tag{5.114}
\]

Substituting Eq. 5.111 into the balance equation 5.109, we transform it for conditions where \( \nu_{0a} = \nu_{0a} = \nu_{0a} \) and the term \( S_0 \) can be neglected:

\[
T_e - T_a = \frac{2 \gamma}{3} \frac{e^2 E^2 \lambda_{0e}}{m_e \kappa_{0a} (\nu_{0a})} \tag{5.115}
\]

At \( \nu' = \nu' = \text{const} \) and \( \kappa' = \kappa' = \text{const} \), the right-hand side of the equality 5.113 is an equation in \( T_e \). Generally, when \( \nu \) and \( \kappa \) depend on the electron velocity and, accordingly, the averaged values \( \nu' \) and \( \kappa' \) (Eq. 5.107) depend on the electron temperature, the equality 5.113 is an equation in \( T_e \). For instance, for \( \nu' = v/v \) with \( \lambda = \text{const} \) and \( \kappa' = \kappa' = \text{const} \), we find, using Eqs. 5.113 and 5.112:

\[
T_e - T_a = \frac{1}{2} \frac{e^2 E^2 \lambda_{0e}}{m_e \kappa_{0a} T_e} \tag{5.116}
\]

and for \( T_e \gg T_a \)

\[
T_e \approx \frac{1}{2} \frac{eE \lambda_{0e}}{\sqrt{6} \kappa_{0a}} \tag{5.117}
\]

When the inequality \( \nu_{0e} \gg \nu_{0e}, \nu_{0a} \) fails one should consider the effect of collisions of electrons with electrons and ions on the function \( f_\parallel \). It is rather difficult to account for electron–electron collisions, because the collision term \( S_0 \) for them cannot be represented as a differential operator, and the equation for \( f_\parallel \) has the integro-differential form. To solve it the function \( f_\parallel \) is expanded in Laguerre polynomials. The calculation result can be represented in a form similar to Eq. 5.111:
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\[ u \approx \frac{\gamma eE}{m_e \nu_{ei}} \]

where \( \gamma = 1.96 \), \( \nu_{ei} \) can be found from Eq. 5.103 with the aid of Eq. 2.69:

\[ \nu_{ei} = \frac{4\sqrt{2\pi}}{3} \frac{ne^4}{m_e^{1.5} T_e^{1.5}} L_e \]  

(5.116)

Substituting Eq. 5.115 into the balance equation 5.109, we obtain the equation for the electron temperature, which is analyzed in Section 7.10.

For a fully ionized plasma the field of application of the results obtained is generally very limited. Firstly, beginning with comparatively low electric field strengths the energy transferred to the electrons from the field does not have enough time to be retransferred to the ions, and the stationary state of balance described by Eq. 5.109 cannot be realized. Secondly, fast electrons switch to the continuous-acceleration regime, with the result that the distribution function becomes strongly anisotropic (see Section 7.11).

We have demonstrated that the electron velocity distribution with a predominant effect of electron-electron collisions is Maxwellian. With a sufficiently high degree of ionization at which the frequency of electron-electron collisions exceeds that of inelastic collisions \( (\nu_{ee} \gg \nu_{in}) \), this distribution can be used for establishing the efficiency of inelastic processes, ionization in particular. By using the approximation equation (Eq. 2.86) for the ionization cross section, we obtain

\[ v' = \int_{v_i}^{\infty} n_s \delta(v') \left( \frac{v^2}{v_i^2} - 1 \right) 4\pi v^2 f_0 dv \]

\[ = 2 \sqrt{\frac{2}{\pi}} n_s \sqrt{\frac{T_e}{m_e}} \left( 1 + 2 \frac{T_e}{v_i^2} \right) \exp \left( -\frac{v_i^2}{T_e} \right) \]  

(5.117)

where \( v_i^2 = 2e \delta/m_e \). For \( \delta \gg T_e \) we find

\[ v' = 2 \sqrt{\frac{2T_e}{\pi m_e}} n_s \delta \exp \left( -\frac{v_i^2}{T_e} \right) \]  

(5.118)

5.7 EFFECT OF MAGNETIC FIELD ON ELECTRON DISTRIBUTION FUNCTION

In the presence of constant electric and magnetic fields the electron distribution function should be sought in the form

\[ f(v) = f_0(v) + \frac{eE}{v} f_{1x}(v) + \frac{eB}{v} f_{1y}(v) + \frac{eB}{v} f_{1z}(v) \]

\[ = f_0(v) + \frac{v}{v} f_{1x}(v) \]  

(5.119)

Assuming, as before, that the distribution function is independent of the time and coordinates, we can write Eqs. 5.22 for the functions \( f_0, f_{1x}, f_{1y}, f_{1z} \) as follows:

\[ \frac{neE}{m_e} \frac{d}{dv} \left( \delta \nu_{ei} \right) = S_0 \]  

(5.120)

\[ \frac{neE dE}{m_e} + \frac{ne}{m_e} \left[ H \times f_0 \right] = -S_i \]  

(5.121)

The second equation is the vector form of the three equations for the relationship between the directed distribution function components \( f_{1x}, f_{1y}, f_{1z} \) and the isotropic component \( f_0 \). The collision term \( S_i \) has the following components:

\[ S_{ik} = \frac{3}{4\pi} \int_{0}^{2\pi} \delta(\theta) \cos \theta \backslash d\theta \]  

(5.122)

where \( \theta \) is the angle between the velocity vector and the \( k \) axis, and the integration is done over the solid angles enclosing all the directions of the velocity vector. Substituting the integral of elastic electron-atom collisions into Eq. 5.122 and assuming the pre- and postcollision electron velocities to be equal, we get

\[ S_{ik} = \frac{3}{4\pi} n_s \int_{0}^{2\pi} \cos \theta \backslash d\theta \int_{0}^{\pi} f_{ik}(v) \sigma d\Omega \]  

(5.123)

The velocity vector change on elastic collisions is related to the scattering angle by Eq. 2.16. Substituting it into the second integral of Eq. 5.123, we find

\[ n_s \int_{0}^{\pi} f_{ik}(v) \sigma d\Omega = -n_s \left( f_{ik} \right) \int_{0}^{\pi} (1 - \cos \theta) \sigma d\Omega \]

\[ = -n_s \left( f_{ik} \right) = \frac{-v' f_{ik}}{v} \]

and, further,

\[ S_{ik} = \frac{3}{4\pi} \frac{n_s v'}{v} \int_{0}^{2\pi} \sum_{i} f_{i} \sigma \cos \theta \backslash d\theta \int_{0}^{\pi} v \cos \theta \backslash d\theta \]
Obviously, the integral over the velocity directions is different from zero only for \( l = k \):

\[
\int_{\omega} v \cos \Theta_k \, d\omega = \delta_{k0} \int \cos^2 \Theta_k \, d\omega = \frac{4\pi}{3} \delta_{k0}
\]

Substituting, we have \( S_{lk} = -nv_{\text{ef}} f_{l0} \), or in vector form,

\[
S_1 = -nv_{\text{ef}} f_1
\]

(5.124)

Similar expressions are obtained for the collision term associated with the inelastic electron-atom collisions (see Section 5.3). Adding them to Eq. 5.124, we find, for \( v_{\text{es}} \gg v_{\text{ew}}, v_{\text{ed}} \), that is, when electron-electron and electron-ion collisions are insignificant:

\[
S_1 = -n(v_{\text{es}} + v_{\text{ew}} + v_{\text{ed}}) f_1 = -nv_{\text{ef}} f_1
\]

(5.125)

Substituting Eq. 5.125 into Eq. 5.121, we get

\[
\frac{eE}{m_e} \frac{df_0}{dv} + \frac{e}{m_e} [H \times f_1] - v_{\text{ef}} f_1 = 0
\]

(5.126)

Let us direct the \( 0z \) axis parallel to \( H \), and the \( 0x \) axis so that the vector \( E \) lies in the \( xz \) plane. Then the projections of Eq. 5.126 on the coordinate axes take the form:

\[
\frac{eE_z}{m_e} \frac{df_0}{dv} - \frac{eH}{m_e} f_{1x} = v_{\text{ef}} f_{1x} = 0;
\]

\[
\frac{eH}{m_e} f_{1y} - v_{\text{ef}} f_{1y} = 0;
\]

\[
\frac{eE_x}{m_e} \frac{df_0}{dv} - \frac{eE_x}{m_e} f_{1z} = v_{\text{ef}} f_{1z} = 0
\]

Solving these equations for \( f_{1x}, f_{1y}, f_{1z} \), we obtain

\[
f_{1x} = \frac{eE_z}{m_e(\omega_H + v^2)} \frac{df_0}{dv};
\]

\[
f_{1y} = \frac{eE_x}{m_e(\omega_H + v^2)} \frac{df_0}{dv};
\]

\[
f_{1z} = \frac{eE_y}{m_e v} \frac{df_0}{dv}
\]

(5.127)

where the following notation is introduced: \( E_z = E_x \) is the component of \( E \) parallel to the magnetic field, \( E_z = E_y \) is the component perpendicular to the magnetic field, and \( \omega_H = eH/m_e c \) is the cyclotron frequency (it is known to determine the angular velocity of electron gyration in the magnetic field (see Section 8.1)).

From Eq. 5.127 it is possible, as before, to obtain general equations for the directed electron velocity. Substituting Eq. 5.127 into Eq. 5.21, we find

\[
u_{1x} = \frac{4\pi}{3} \frac{eE_z}{m_e} \int_0^\infty \frac{v^3}{(\omega_H + v^2)} \frac{df_0}{dv} dv;
\]

\[
u_{1y} = \frac{4\pi}{3} \frac{eE_x}{m_e} \int_0^\infty \frac{v^3\omega_H}{(\omega_H + v^2)} \frac{df_0}{dv} dv;
\]

\[
u_{1z} = \frac{4\pi}{3} \frac{eE_y}{m_e} \int_0^\infty \frac{v^3}{v} \frac{df_0}{dv} dv
\]

(5.128)

For \( v = \text{const} \), integration can be done by parts. Then Eq. 5.128 is independent of the form of \( f_0 \):

\[
u_{1x} = \frac{eE_z}{m_e} \frac{v}{(\omega_H + v^2)};
\]

\[
u_{1y} = \frac{eE_x}{m_e} \frac{\omega_H}{(\omega_H + v^2)};
\]

\[
u_{1z} = -\frac{eE_y}{m_e v}
\]

(5.129)

As can be seen, the magnetic field does not affect the velocity component \( u_x \) parallel to it. This is only natural, since the Lorentz force is proportional to the vector product of the field by the velocity and does not have a component parallel to \( H \). The effect of the magnetic field on the perpendicular velocity components is substantial at \( \omega_H > v \). Then the velocity in the direction of the component \( E \) perpendicular to \( H \) \( \left( v_x \right) \) decreases, and a velocity component perpendicular to \( E \) and \( H \) appears. This effect can be explained by gyration of electrons in a plane perpendicular to \( H \). At \( \omega_H < v \) the gyration does not have a chance to manifest itself, since during the intercollisional time the electrons move practically along a straight line. At \( \omega_H > v \) the electrons make many revolutions between collisions. As a result the periods of electron acceleration under the effect of \( E_z \) (during which \( vE_z < 0 \)) alternate with slowing-down periods (when \( vE_z > 0 \)), and the summary efficiency of electron acceleration in the direction of \( E_z \) drops off. A detailed analysis of the effect of the magnetic field on the direction of the electron motion is given in Chapter 9.

Using the obtained expressions for the components \( f_1 \), we consider
the effect of the magnetic field on \( f_0 \). In the equation for \( f_0 \) (Eq. 5.120), the magnetic field affects the first term, which defines the energy acquisition by electrons under the effect of the electric field. This term, as noted in Section 5.4, represents the divergence of a spherically symmetric flux in the velocity space

\[
div v \Gamma_{\text{eff}} = \frac{1}{v^3} \frac{d}{dv} (v^4 \Gamma_{\text{eff}})
\]

where

\[
\Gamma_{\text{eff}} = -\frac{ne}{3e} \left( E \frac{1}{3m_e} \ln \frac{E}{3m_e} + E f_{1e} + Ef_{2e} \right)
\]

Substituting Eq. 5.127 into Eq. 5.130, we get

\[
\Gamma_{\text{eff}} = -\frac{ne^2}{3m_e^2} \left( \frac{E^2}{\nu} + \frac{E^2}{\omega_H + \nu} \right) \frac{df_0}{dv}
\]

Since the flux density is proportional to the derivative \( df_0/dv \), the flux is of a diffusion nature and is caused by accidental "straying" of electrons on the velocity scale. As shown in Section 5.4 the diffusion coefficient is determined by the mean-square change in electron velocity within the intercollisional time (Eq. 5.60): \( D_v = \frac{1}{2} (\Delta v)^2 \tau^{-1} = \frac{1}{2} (\Delta v_E)^2 \nu^{-1} \)

In the absence of a magnetic field the velocity change is associated only with acceleration in the electric field \( (\Delta v_E = eE/m_e\nu) \); therefore the flux value has the form of Eq. 5.59. At \( E \parallel H \) the value of \( \Delta v_E \) is the same and it remains practically the same at \( E \perp H \) when \( \omega_H < \nu \). But at \( \omega_H > \nu \) the electron gyration leads, as noted above, to alternating periods of acceleration and slowing down of the electrons. Each acceleration period \( \Delta t_\text{el} \) is obviously equal to the time of half-revolution of an electron \( \Delta t_E = \pi/\omega_H \). Accordingly, the maximum velocity change during the intercollisional time is \( \Delta v_E \approx (E f_1/m_e) \), \( \Delta t_E \approx \pi eE_1/m_e\omega_H \). It is this change that determines the diffusion coefficient for \( \omega_H > \nu \) with an accuracy to a factor of the order of unity:

\[
D_v = \frac{(\Delta v_E)^2}{6} = \frac{e^2E_1^2}{m_e\omega_H^2} \nu
\]

The equation obtained corresponds to the second term in Eq. 5.131. The expression 5.131 for \( \Gamma_{E} \) can be written in the form 5.59 by introducing an effective electric field:

\[
E_{\text{eff}}^2 = E_1^2 + E_1^2 \frac{\nu^2}{\omega_H^2 + \nu^2}
\]

Then Eq. 5.120 for \( f_0 \) will be formally the same as in the absence of a magnetic field; the dependence of \( f_0 \) on \( H \) appears in it only in terms of \( E_{\text{eff}} \). Therefore, to determine the effect of the magnetic field on \( f_0 \) we can use the overall results of Sections 5.4-5.6, replacing \( E \) by \( E_{\text{eff}} \). When establishing the specific form of \( f_0 \) we must remember that \( E_{\text{eff}} \) in Eq. 5.132 generally depends on \( \nu \).

For \( \nu = \text{const} \) there is no such dependence, and we can use the equations of Sections 5.4-5.6 directly. Thus when only elastic collisions are substantial the distribution \( d_0 \) is Maxwellian. The electron temperature is then obtained from Eq. 5.66, in which \( E \) must be replaced by \( E_{\text{eff}} \) as in Eq. 5.132:

\[
T_e = T_s + \frac{2}{3} \frac{e^2}{m_e\nu^2} \left[ E_{\text{eff}}^2 + \frac{\nu^2}{\omega_H^2 + \nu^2} E_1^2 \right]
\]

Interestingly, in an electric field perpendicular to the magnetic field, at \( \omega_H \gg \nu \) the function \( f_0 \) is altogether independent of \( \nu \). Indeed, in accordance with Eq. 5.131 the rate of energy accumulation by electrons, which is determined by \( \Gamma_{\text{el}} \), is proportional to \( \nu \), as is the collision term due to elastic collisions. Therefore in Eq. 5.120 the collision frequency cancels out, and it takes the form

\[
\frac{1}{3} \frac{e^2E_1^2}{m_e\omega_H^2} \frac{df_0}{dv} = \frac{1}{2} \frac{\nu}{e} f_0 + \frac{T_e}{m_e} \frac{df_0}{dv}
\]

The solution of this equation for \( \kappa_\alpha = \text{const} \) acquires the form of a Maxwellian distribution with a temperature

\[
T_e = T_s + \frac{1}{2} \frac{e^2E_1^2}{m_e\omega_H^2}
\]

As indicated in Section 5.6, with a predominant effect of electron-electron collisions on the function \( f_0 \) (i.e., at \( \nu_\alpha \gg \kappa_\alpha \nu_\alpha \)) it is near-Maxwellian. The electron temperature is then found from the energy balance equation 5.109. The magnetic field affects the left-hand side of the balance equation. It includes the directed electron velocity, whose components can, at \( \nu_e \ll \nu_\alpha \), be obtained from Eq. 5.128. Then the balance equation can be reduced to a form similar to Eq. 5.113:

\[
T_e - T_s = \frac{2\gamma}{3} \frac{e^2}{m_e\kappa_\alpha^2} E_{\text{eff}}^2
\]

where \( E_{\text{eff}}^2 = E_1^2 + \xi(\nu^2/\omega_H^2 + \nu^2)E_1^2 \). Here \( \nu \) and \( \kappa \) are summary averaged values yielded by Eq. 5.107 and generally depending on \( T_e \); and \( \gamma \) and \( \xi \) are numerical factors of the order of unity, which are determined by the dependence of \( \nu_\alpha \) and \( \kappa_\alpha \) on \( \nu \), with \( \xi \) additionally
The large difference between the relaxation times of the isotropic and directed components of the distribution function \((\tau_0, \tau_1)\) is obviously due to the difference in the effect of collisions on the value and direction of the electron velocity.

By comparing the field alternation period \(T\) with the relaxation times \(\tau_0\) and \(\tau_1\), we can isolate the low-frequency case, for which \(T\) greatly exceeds \(\tau_0\) and, all the more so, \(\tau_1\); that is, the following inequalities hold:

\[
T \gg \frac{1}{\nu f_0}, \quad \omega \ll \nu f_0
\]

(5.138)

where \(\omega = 2\pi f/\nu\) is the angular frequency of field alternation. The inertial terms in both equations 5.137 are small compared with the collision terms, and they can be neglected, which means that the functions \(f_0\) and \(f_1\) are the same as in a constant magnetic field and are determined at each instant by the instantaneous field value.

For high frequencies, when the inequalities opposite to Eqs. 5.138 are fulfilled,

\[
T \ll \frac{1}{\nu f_0}; \quad \omega \gg \nu f_0
\]

(5.139)

that is, the field alternation period is much less than the relaxation time of \(f_0\); this function cannot catch up with the field alternation. Here \(f_0\) is nearly constant in time and is only slightly modulated by alternations with a frequency equal to the doubled frequency of the field. It is easy to ascertain with the aid of Eqs. 5.137 that the depth of this modulation is of the order of \(\nu f_0/\omega\). When considering the high-frequency case (Eq. 5.139) we neglect this modulation and assume \(f_0\) to be time invariant, which simplifies the solution of the set of equations 5.137.

Consider first the equation for \(f_1\). Assume that the electric field alternates according to the harmonic law

\[
E = E_0 \cos \omega t
\]

(5.140)

The collision term \(S_1\) is represented in the form 5.28: \(S_1 = -\nu f_1 e^\nu f_1\), assuming that \(\nu_0 \gg \nu_\text{el}, \nu_\text{el}\). Then

\[
\frac{\partial f_1}{\partial t} + \nu f_1 = \frac{eE_0}{m_e} \frac{\partial f_0}{\partial v} \cos \omega t
\]

(5.141)

The stationary solution of Eq. 5.141 can be sought in the form

\[
f_1(v, t) = f_{1\text{st}}(v) \cos \omega t + f_{1\text{st}}(v) \sin \omega t
\]

(5.142)

where \(f_{1\text{st}}\) and \(f_{1\text{st}}\) are time invariant. Substituting Eq. 5.142 into 5.141, we get

\[
(af_{1\text{st}} + vf_{1\text{st}}) \cos \omega t - (af_{1\text{st}} - vf_{1\text{st}}) \sin \omega t - \frac{eE_0}{m_e} \frac{\partial f_0}{\partial v} \cos \omega t = 0
\]
Equating the coefficients at $\cos \omega t$ and $\sin \omega t$ separately to zero, we find the expressions for $f_1$ and $f_2$:

$$f_1 = \frac{eE_0\nu}{m_e(\omega^2 + \nu^2)} \frac{\partial f_0}{\partial \nu}, \quad f_2 = \frac{eE_0\omega}{m_e(\omega^2 + \nu^2)} \frac{\partial f_0}{\partial \nu} \quad (5.143)$$

These expressions define the directed electron velocity in a high-frequency electric field. Substituting Eqs. 5.142 and 5.143 into Eq. 5.13, we obtain

$$u = u_1 \cos \omega t + u_2 \sin \omega t;$$

$$u_1 = \frac{4\pi eE_0}{3 m_e} \int_0^\infty \frac{\nu \nu^3}{\omega^2 + \nu^2} \frac{\partial f_0}{\partial \nu} \, d\nu; \quad (5.144)$$

$$u_2 = \frac{4\pi eE_0}{3 m_e} \int_0^\infty \frac{\omega \nu^3}{\omega^2 + \nu^2} \frac{\partial f_0}{\partial \nu} \, d\nu$$

At $\nu = \text{const}$ the integrals appearing in Eqs. 5.144 become, after integration by parts, normalization integrals, and

$$u_1 = \frac{eE_0}{m_e} \frac{\nu}{(\omega^2 + \nu^2)}; \quad u_2 = \frac{eE_0}{m_e} \frac{\omega}{(\omega^2 + \nu^2)} \quad (5.145)$$

These equations yield two components of the directed velocity. The first one $u_1$ characterizes the electron motion in phase with the field (it changes with time in the same way as the field), and the second $u_2$, the motion shifted in phase by $\pi/2$ (the time dependence is determined by $\sin \omega t$ instead of by $\cos \omega t$). At $\omega \ll \nu$ Eqs. 5.144 and 5.145 for the component $u_1$ change to the previously obtained expressions for the directed velocity in a constant field (Eqs. 5.56 and 5.57) and the component $u_2$ is much less than $u_1$. The condition $\omega \ll \nu$ corresponds to a large number of electron collisions during a field period, that is, to a small change in field during the intercollisional time. Naturally, the relationship between the directed velocities and the field is then the same as in a constant field. At $\omega \gg \nu$ the directed velocity component $u_1$, which is in phase with the field, is sharply reduced, whereas the component $u_2$, which is shifted in phase by $\pi/2$, becomes much larger than $u_1$. This is because within the intercollisional time many field periods take place, during which the electron accelerates and slows down alternately. Such a “reactive” energy exchange precisely corresponds to the velocity-field phase shift, which is close to $\pi/2$.

Note that Eqs. 5.144 and 5.145 for $u_1$ and $u_2$ coincide with the equations for the velocity components $u_z$ and $u_y$ in a constant electric field ($E \parallel Ox$) perpendicular to the magnetic field if we replace $\omega$ by $\omega t$ (see Eqs. 5.128 and 5.129 at $E_{11} = 0$). This coincidence reflects the identical, periodic nature of the effect of the electric field on the electrons. (In the presence of a constant magnetic field it is associated with electron gyration, and in a variable electric field, with the alternation of the field itself.)

Let us now find $f_0$. To this end we substitute the expression for $f_1$, determined by Eqs. 5.142 and 5.143, into the equation for $f_0$ (Eq. 5.137). Then it takes the form

$$\frac{\partial n f_0}{\partial t} = -\frac{1}{6} \frac{n e^2 E_0^2}{m_e^2 \nu^2} \left[ \left( 1 + \cos(2\omega t) \right) \frac{\partial}{\partial \nu} \left( \frac{\nu^3}{\omega^2 + \nu^2} \frac{\partial f_0}{\partial \nu} \right) \right. \quad (5.146)$$

$$+ \sin(2\omega t) \frac{\partial}{\partial \nu} \left( \frac{\omega^3}{\omega^2 + \nu^2} \frac{\partial f_0}{\partial \nu} \right) \right] = S_0$$

As has already been noted for $\omega \gg \nu$ $f_0$ is modulated only slightly and can be assumed constant to a first approximation. We obtain the equation in this approximation by averaging Eq. 5.146 over the alternation period:

$$\frac{1}{6} \frac{n e^2 E_0^2}{m_e^2 \nu^2} \int \frac{\nu^3}{\omega^2 + \nu^2} \frac{\partial f_0}{\partial \nu} \, d\nu = S_0 \quad (5.147)$$

In the next approximation we can find the variable addition to $f_0$ with the aid of Eq. 5.146; it can be seen that this addition varies with a frequency of $2\omega$.

The left-hand side of Eq. 5.147 defines the energy acquired by the electrons due to their motion in the electric field. As before, it represents the divergence of a spherically symmetric flux

$$\Gamma_{ee} = -\frac{1}{6} \frac{n e^2 E_0^2}{m_e^2 (\omega^2 + \nu^2)} \frac{\partial f_0}{\partial \nu} \quad (5.148)$$

which can be described as the result of the diffusion, and in velocity space

$$\Gamma_{ee} = -D_e \frac{\partial f_0}{\partial \nu}, \quad D_e = \frac{1}{6} \frac{n e^2 E_0^2}{m_e^2 (\omega^2 + \nu^2)} \quad (5.149)$$

At $\omega \ll \nu$, when the field changes only slightly during the intercollisional time, the nature of electron acceleration is the same as in a constant field. Therefore the equation for the diffusion coefficient $D_e$ at $\omega \ll \nu$ is the same (see Eq. 5.59); the quantity $E^2$ appearing in it is averaged over the field period $(E^2 = \langle E^2 \rangle/2)$. At $\omega \gg \nu$ the periods of electron acceleration and slowing down alternate because of the repeated changes in the direction of the field during the intercollisional time. The maximum velocity increment in a field of frequency $\omega$ is equal to $(\Delta v)_{max} = \frac{\pi E_0}{m_e \nu}$.
Then the coefficient of diffusion in velocity space (see Eq. 5.60),

\[ D_\phi = \frac{1}{2} \frac{e^2 E_0^2 \nu}{m_{\phi} \omega_{\phi}} \]

corresponds to Eq. 5.149.

The relation 5.149 for the flux density \( \Gamma_{\phi z} \) can be represented in a form similar to Eq. 5.59 if we introduce the effective field

\[ E_{\text{eff}} = \frac{1}{2} \frac{e^2 E_0^2}{\nu + \omega_{\phi}} \]

(5.150)

Therefore the results of determination of \( f_0 \) for a constant electric field can be extended to the case of a high-frequency field. The situation here is precisely the same as for coexisting constant electric and magnetic fields (see Section 5.7). Moreover, because of the identical nature of the acceleration the equations for \( f_0 \) are themselves similar in the two cases (c.f. Eqs. 5.150 and 5.132 for \( E_i = 0 \)). Therefore we do not revert to the application of Eq. 5.150 in different conditions, but give the expression for \( f_0 \) when only elastic electron-atom collisions are substantial. In accordance with the above we can use Eq. 5.63, replacing \( E_0 \) by \( E_{\text{eff}} \). Then we obtain

\[ f_0 = A \exp \left( - \frac{m_{\phi} \nu}{T_0 + \frac{1}{2} \frac{e^2 E_0^2}{m_{\phi} \omega_{\phi} (\omega^2 + \nu^2)}} \right) \]

(5.151)

If we neglect \( T_0 \) (the strong-field case), Eq. 5.151 acquires the form

\[ f_0 = A \exp \left[ - \frac{3 m_{\phi} \omega_{\phi}}{e^2 E_0^2} \left( \omega^2 \nu^2 + 2 \int_0^\infty \nu v^2 \text{d}v \right) \right] \]

(5.152)

At \( \nu = \text{const} \) Eq. 5.151 leads to a Maxwellian velocity distribution with a temperature

\[ T_\phi = T_o + \frac{e^2 E_0^2}{3 m_{\phi} \omega_{\phi} (\omega^2 + \nu_{\phi}^2)} \]

(5.153)

At sufficiently high frequencies (\( \omega \gg \nu \)) the distribution will be Maxwellian for any dependence \( \nu(\nu) \). Then the electron temperature is altogether independent of \( \nu \):

\[ T_\phi = T_o + \frac{e^2 E_0^2}{3 m_{\phi} \omega_{\phi} \omega^2} \]

5.9 Ion Distribution Function in Electric Field

As shown in Section 5.1 the ion distribution function in a weakly ionized plasma, at electric fields much less than critical (Eq. 5.5), \( E \ll E_{\text{pi}} = \frac{T(e\lambda_{\text{pi}}) \text{eE}_0}{m_{\phi} \omega_{\phi}} \), is near-equilibrium (near-Maxwellian). The deviation of the distribution from equilibrium can be found by successive approximations, which makes it possible to obtain the directed velocity, the momentum and energy fluxes, the correction to the average energy, and other characteristics associated with the electric field (see Chapters 6 and 7). As the electric field approaches the critical value the ion distribution function increasingly deviates from equilibrium. The nature of this deviation depends on the ion-atom collisions. Since their masses are similar, and energy transfer as well as momentum transfer in collisions may be considerable, the directed ion velocity in a strong electric field is commensurate with the random velocity or exceeds it; that is, the distribution function is strongly anisotropic. Therefore, the method for solving the kinetic equation by expanding it in degrees of anisotropy, which was described in the earlier sections, is inapplicable for ions. For this reason the problem of determining the distribution function for ions in an electric field is much more complicated than for electrons, and it can be solved only in cases permitting simplification.

The distribution function can be found, for instance, for the "relay" model, which describes approximately the ion motion in a weakly ionized plasma containing atoms and ions of the same species at electric field strengths greatly exceeding the critical value. Here the principal role is played by ion-atom collisions of the type of resonance charge exchange, which are accompanied by insignificant energy transfer. Each such collision results in disappearance of the ion accelerated by the electric field in the intercollisional period (it turns into a neutral atom) and in the appearance of a slow ion with an energy practically equal to that of the neutral atom from which it was produced. Taking into account only such collisions, one can use, for approximate consideration, a model in which each collision reduces the ion energy to zero. Within the framework of this model the ion velocity depends exclusively on the acceleration due to the electric field in the period between two collisions. Accordingly, the directions of the motion and of the electric field coincide and the velocity distribution is determined by that of the intercollisional time.

To find the distribution function we can make use of the kinetic equation, which takes the following form for the model at hand:

\[ \frac{e\text{E}\text{d}f_i}{m_i \text{d}v} = \frac{\delta f_i}{\delta t} \]

(5.155)

where \( f_i \) depends on just one velocity component parallel to the electric field. In accordance with the model used the collision term must include only charge exchange collisions, which lead to the removal of ions from
DISTRIBUTION FUNCTION OF CHARGED PARTICLES

the given velocity range. Therefore it can be written thus:

\[ \frac{\delta f_i}{\delta t} = -v_{nu,5a}(v)f_i(v) = -v_{u}(v)f_i(v) \]  

(5.156)

where \( s_{5a} \) is the total charge exchange cross section, and \( v_{nu} = n_{5a}v_{5a} \) is the frequency of charge exchange collisions. We assume that the relative velocity on collision is practically equal to the ion velocity (that the velocity of the neutral atoms is much less than that of the ions). Substituting Eq. 5.156 into Eq. 5.155, we obtain \( (eE/m_i)\, df/dv = -v_{nu}(v)f_i \). Integrating this equation yields the ion velocity distribution function along the field

\[ f_i(v) = A \exp\left(-\frac{m_i}{eE} \int_0^v v_{nu}(v) \, dv \right) \]  

(5.157)

where \( A \) is the normalization factor, which is found from the condition

\[ \int_{-\infty}^{\infty} f_i(v) \, dv = 1 \]  

(5.158)

Let us now consider a highly ionized plasma in which

\[ \nu_0 \gg \nu_{nu}; \quad \eta \gg \frac{\nu_{nu}}{s_{5a}} \]  

(5.162)

The role of the various collisions can be evaluated with the aid of the kinetic equation as in Section 5.6 for electrons. For ions in a homogeneous plasma it has the form

\[ \frac{neE}{m_i} \text{ grad } f_i = S_{5a} + S_{5e} + S_{5a} \]  

(5.163)

The balance of the average momentum and average energy depends on the first three terms of the equation: the first term characterizes the ion acceleration in the electric field, and the second and third, the momentum and energy exchange with electrons and neutral atoms. Therefore the term proportional to the field is of the same order as the sum of the first two collision terms. Ion–ion collisions obviously do not affect the average momentum or the average energy of the ions. They only lead to momentum and energy redistribution among the ions. At the same time, in the case under consideration the ion–ion collision term \( S_{ii} \) greatly exceeds the other terms in Eq. 5.163. Because of the small energy and momentum exchange on ion-electron collisions it is usually much greater than the ion–electron collision term:

\[ \frac{S_{ii}}{S_{5e}} > \frac{v_{ii}v_{5e}}{2 \nu_{ii}} \]  

If the inequality 5.162 holds, it is also much greater than the ion–atom collision term \( S_{5i}/S_{5a} = v_{ii}/v_{5a} \gg 1 \). Therefore the form of the distribution function is determined, to a first approximation, by the vanishing of the ion–ion collision term. As shown in Section 4.2, this condition leads to the Maxwellian distribution:

\[ f_i(v) = \left( \frac{m_i}{2\pi T_i} \right)^{3/2} \exp\left[-\frac{m_i(v - u_i)^2}{2T_i}\right] \]  

(5.164)

The directed velocity \( u_i \) and the ion temperature \( T_i \) can be defined by the average-momentum and average-energy balance equations. They are obtained by multiplying the kinetic equation 5.163 by the momentum \( m_i v \) or by the energy \( m_i v^2/2 \), respectively, and term-by-term integration over the entire velocity space. This procedure is carried out for a more general case in Sections 6.3 and 6.4. We give here the corresponding equalities only for conditions where collisions of ions with electrons play an insignificant role compared with their collisions with neutral atoms, that is, at \( S_{5e}/S_{5a} = (m_i/m_e)\nu_{5a}/\nu_{ii} \ll 1 \). Under such conditions the
momentum balance amounts to the equality of the electric force and the average momentum transferred on ion-atom collisions: \[ eE = \frac{\mu a v_a u_i}{} \] This equality gives the directed ion velocity
\[ u_i = \frac{eE}{\mu a v_a} = \frac{2eE}{m v_a} \] (5.165)
at \[ m_i = m_e \]. The ion energy balance equation amounts to the equality of the average energy acquired by ions in the electric field and the average energy lost by them in collisions with neutral atoms: \[ eEu_i = \frac{\kappa a v_a (T_i - T_n)}{} \]
where \[ \kappa a = 2m m_u (m_i + m_e)^2 = \frac{1}{2} \] at \[ m_i = m_e \]. It results in the following relation for the ion temperature:
\[ T_i - T_n = \frac{e^2 E^2}{m v_i} \] (5.166)

At higher degrees of ionization, when the effect of electron-ion collisions is substantial, one should include in the energy balance of the ions not only their energy exchange with neutral atoms, but also the energy exchange between electrons and ions.

6

DISTRIBUTION FUNCTION MOMENTS EQUATIONS

6.1 DISTRIBUTION FUNCTION MOMENTS

Under given external conditions the kinetic equations make it possible, in principle, to find particle distribution functions and thus obtain the macroscopic characteristics of the plasma. In view of the complexity of kinetic equations it is not always possible to find their complete solution. Many problems, however, are solved using approximate equations for the distribution function moments, which can be derived from kinetic equations. The distribution function moments are combinations of particle velocity components averaged over the distribution; they may be linear (first-order moments), quadratic (second-order moments), and so on. Let us establish the physical quantities determined by these moments. Three first-order moments represent the average values of the velocity components.*

\[ u_i = \langle v_i \rangle = \int f(v) d^3v \] (6.1)

where \[ u_i \] and \[ v_i \] are the projections of the vectors \[ u \] and \[ v \] on one of the coordinate axes. The equalities 6.1 can also be written in vector form:
\[ u = \int f(v) d^3v \]
The total velocity of each particle can be represented as the sum
\[ v = u + w \] (6.2)

*Hereafter, in equations that can be applied to particles of different types we omit, for brevity, the suffix denoting the type of particle.