Finally we turn to the continuous part of the beta spectrum; this no longer has the form of discrete energy lines with which we were so familiar from the study of the atomic systems, and which we also encountered in some nuclear spectra. If we wish to retain the scheme of nuclear energy levels, then the electrons must be carrying away only part of the energy of the nuclear transition, the remainder being taken away by one or more other quanta.

Indeed, we know today that an electron and a neutrino are simultaneously emitted and share the energy of the nuclear transition, while momentum is balanced between the nucleus itself and the electron and neutrino. The neutrino is a noncharged, weakly interacting particle with very small or zero mass. Because of this sharing process, and since the mass of the neutrino is negligible, the upper end of the continuous beta spectrum should give the energy difference between the initial and final state of the transition; this corresponds to $0.522 \pm 0.015$ MeV, and is indicated in the level diagram of Fig. 2.17. Transitions involving the emission of an electron (and neutrino) are shown by arrows slanting to the right (increase in nuclear $Z$); positron emission is shown by arrows slanting to the left. We usually write this as

$$A (Z) \rightarrow A (Z + 1) + \beta^- + \nu$$

(5.7)

In Eq. 5.7 $A$ and $Z$ stand for the atomic number and charge of the nucleus, while $\beta^-$ is the symbol for the electron and $\nu$ for the neutrino.

Thus we have seen that simple quantum-mechanical systems, such as the free hydrogen atom or other free atoms, or even nuclei, are characterized by discrete energy levels. The systems can undergo transitions between these levels with the emission or absorption of energy quanta, in the form of photons, electrons, or other particles. When only one particle is emitted, it carries away the whole energy of the transition; otherwise the energy is shared between the participating particles. The energy of these particles can be measured, as was shown, by various techniques, and thus provide information on the structure of the energy levels of these systems. For a more sophisticated treatment of atomic spectroscopy the reader should consult Chapter 7 and the references therein.

1. Introduction

In this chapter systems with very large numbers of interacting particles will be considered. Actually most of matter as it can be perceived with our senses belongs in this category. In matter in the gaseous state, the distances between molecules are large and therefore the forces are weak. In solids, however, the forces are much stronger. Understanding of the thermodynamic properties of "bulk" matter, based on the microscopic behavior of the constituent molecules or atoms, was first achieved through the statistical mechanics developed by Boltzmann. Because of the immense number of interacting bodies, the statistical approach is quite valid and has proved highly successful. Classical statistical mechanics, however, was unable to explain several phenomena until quantum mechanical principles were incorporated. According to our current ideas, particles with half-integral spin—such as the electrons—obey so-called "Fermi-Dirac" statistics, while particles with integral spin—such as photons and helium atoms—obey "Bose-Einstein" statistics. The fundamental distinction is that the former type of particles must have a completely antisymmetric
wave function, whereas the latter ones must have a symmetric wave function; this leads to a different distribution function for the probability that a particle will occupy a certain cell in phase space. Since the electronic properties of solids are determined by the behavior of their electrons, it is Fermi statistics that are relevant and will be applied to the description of thermionic emission and the properties of semiconductors.

Most solid-state materials, however, have a crystalline structure; that is, the atoms form a periodic lattice. Advantage can be taken of this periodicity so that the macroscopic behavior of the crystal is predicted from the general parameters of the lattice and the atoms that form it. It is found that the free electrons, instead of occupying distinct energy levels—as they do in the simpler quantum-mechanical systems—are contained in certain energy bands. Knowledge of the “band structure” is necessary in most considerations of the solid state and specifically in the understanding of the behavior of semiconductors. The motion of the free electrons or holes (contained in the valence band) through the lattice can be studied in terms of a single-particle approach. Such phenomena as scattering and the absorption or emission of vibrational quanta (phonons) are invoked and are useful in explaining further details in the macroscopic behavior of the sample.

An experiment in which the resistivity and Hall effect of germanium are measured as a function of temperature is described in Section 3. In Section 4 is given a brief sketch of junction theory and the principles involved; it is relevant to one of the most important applications of semiconductors, the transistor.

1.1 THE FERMI-DIRAC DISTRIBUTION

Let us consider a large ensemble of free Fermi particles (such as electrons); the assumption is made that in phase space† there exist many states that these electrons can occupy. Each “cell” has a phase-space volume of \( h^3 \) (where \( h \) is again Planck’s constant), so that the number of available cells for a differential volume of phase space is

\[
n = \frac{2}{V_0} \int_0^n \int_0^n \int_0^n \frac{2}{h^3} \, dp_x \, dp_y \, dp_z \, \frac{1}{dw_i}
\]

Further, we can obtain the number of states per unit volume per unit energy interval

\[
n_i = \frac{\frac{n'}{dw_i}}{2} = \frac{2}{h^3} \frac{4\pi p_i^2}{2m} \, dp_i \, \frac{1}{dw_i}
\]

and since

\[
w_i = \frac{p_i^2}{2m} \quad dw_i = \frac{2p_i \, dp_i}{2m}
\]

Equation (1.2), which was obtained from very simple considerations, represents the number of states per unit volume per unit energy interval (at a given energy) and is called the “energy density of states.” We note that for a simple ensemble of free Fermi particles (a) all energies are permissible (since \( dN(w)/dw \) is a continuous and not singular function); namely, the energy is not quantized and (b) the number of states increases with increasing energy.

Proceeding further to specify our system, we would like to know which of these infinitely many states are occupied, or in a statistical fashion, what is the probability that a state \( i \) of given energy \( w_i \) be occupied. This is the Fermi-Dirac distribution and is given by

\[
\frac{N_i}{2n} = \left[ \exp \left( \frac{w_i - w_F}{kT} \right) + 1 \right]^{-1}
\]

where

\[
\begin{align*}
&k = \text{the Boltzmann constant}, \\
&T = \text{the temperature of the system, and} \\
&w_F = \text{a characteristic energy, called the Fermi energy or Fermi-level energy.}
\end{align*}
\]

It is interesting to note the properties of this function, graphed in Fig. 3.1:

(a) It is properly bounded, so that it can represent a probability

\[
0 < \frac{N_i}{2n} < 1
\]
3. QUANTUM-MECHANICAL SYSTEMS

Fig. 3.1 Probability of occupancy of a state of energy \( w_i \) as derived from Fermi-Dirac statistics.

(b) For large values of \( w \) it assumes the form of the Boltzmann distribution

\[
\text{Const} \times \exp \left( -\frac{w}{kT} \right)
\]

(c) For \( T = 0 \) it is a step function, with

\[
\begin{align*}
N_{\frac{3}{2}n} &= 1 & w_i &< w_p \\
N_{\frac{1}{2}n} &= 0 & w_i &> w_p
\end{align*}
\]

(d) For \( T \neq 0, w_p \) has the property that \( N(w_p) = \frac{1}{2} \), and as many states above \( w_p \) are occupied, that many states below \( w_p \) are empty.

(e) In solids and for average \( T \neq 0 \), the distribution function is only slightly modified from its shape at \( T = 0 \) (for solids \( w_p \) is of the order of a few electron volts, while \( 1/kT \approx 40 \) at \( T = 300 \)).

Combining the Fermi-Dirac distribution (Eq. 1.3) with the energy density of states (Eq. 1.2) it is possible to obtain any desired distribution. For example, the number of electrons per unit volume (density) at an energy \( w \) in the interval \( dw \) is given by

\[
N(w)dw = \frac{8\pi}{\hbar^2} \sqrt{2\pi m^2_w} \left\{ \exp \left( \frac{w_p - w}{kT} \right) + 1 \right\}^{-1} dw \quad (1.4)
\]

If we express Eq. 1.4 in terms of the cartesian coordinates of the velocity, \( v_x, v_y, \) and \( v_z \), and integrate over \( v_x \) and \( v_z \), we obtain the number of electrons per unit volume with a given velocity in the \( z \) direction, \( v_z \) (in the interval \( dv_z \)). The result of this integration is†

\[
N(v_z)dv_z = \frac{8\pi}{\hbar^2} \frac{m^2_k T}{2} \ln \left\{ 1 + \exp \left( \frac{w_p - mv_z^2/2}{kT} \right) \right\} dv_z \quad (1.4a)
\]

The two distributions given by Eqs. 1.4 are shown in Fig. 3.2.


Even though the majority of the electrons in a solid are not free (as we originally assumed) Fermi-Dirac statistics are applicable, especially to metals. In metals at least one electron per atom has several states available (is in the conduction band), so that it can be considered free; since there will be \( 6 \times 10^{23} \) free electrons per gram mole, statistical methods are well justified.

1.2 ELEMENTS FROM THE BAND THEORY OF SOLIDS

Up to now, no account has been taken of the interatomic or intramolecular forces that might act on the free electrons. Indeed, we expect (from previous experience) that the consideration of some potential in the region where the electrons move will result in the appearance of energy levels; however, because of the periodic structure of this potential, instead of energy levels, energy bands appear, and only the states contained in these bands can be occupied (with any significant probability). In the following paragraphs we will sketch two approaches toward the understanding of the physical origin of the energy bands.

Consider first the one-dimensional problem† of an electron moving in a potential consisting of an infinite sequence of "square" wells of depth \( V_b \), width \( b \) and spaced at a distance \( l \) from one another (Fig. 3.3). The solution of the Schrödinger equation for such a potential gives for the electron a wave function:

\[
\psi_k = u_k(x) e^{ikx} \quad (1.5)
\]

with \( k = 2\pi/\lambda = p/\hbar \) the wave number of the electron. This wave function

consists of the plane wave part $e^{i \omega x}$, and $u_k(x)$, which have the must periodicity of the lattice, namely, $u_k(x \pm l) = u_k(x)$. If there are $N$ lattice sites, the length of the crystal is $Nl$ and we impose the following periodic boundary condition, $\psi_k(x + N) = \psi_k(x)$. This leads to $e^{ikNl} = 1$, or

$$kNl = n2\pi$$

$$k = n2\pi/Nl \quad n = 0, \pm 1, \pm 2, \cdots$$  \hspace{1cm} (1.6)

Equation 1.6 determines the allowed values of $k$, which form almost a continuum because of the very large integer value of $N$. Note that for $N = 1$ one obtains the familiar “particle in a box” energy levels, with

$$E = \frac{p^2}{2m} = \frac{k^2\hbar^2}{2m} = \frac{n^2\hbar^2}{2ml^2}$$

Having determined the wave function, it is possible to solve the Schrödinger equation for the energy eigenvalues

$$H\psi_k = E(k)\psi_k \quad \text{or} \quad \psi^*_kH\psi_k = E(k)$$  \hspace{1cm} (1.7)

where $H$ is the one-dimensional Hamiltonian operator.

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

and $V(x)$ is now the potential of Fig. 3.3.

The solution of Eq. 1.7 is given in graphical form in Fig. 3.4. We note the following:

(a) Even though all values of $k$ are allowed, discontinuities arise at $k = n\pi/l$ (note that for this particular electron wavelength, Bragg reflection from the lattice will occur with a half-angle $\theta = 90^\circ$; $n\lambda = 2l\sin\theta$, hence $\lambda = 2l/n$, and since $\lambda = 2\pi/k$, it follows $k = n\pi/l$).

(b) Not all values of the energy are allowed, but only certain “bands”; other bands of energy are forbidden.

(c) The relation between $E$ and $p$ (or $k$) is no longer the familiar parabolic

$$E = \frac{p^2}{2m} = \frac{k^2\hbar^2}{2m}$$  \hspace{1cm} (1.8)

We can, however, retain this relation if the mass $m$ is assumed variable and a function of $k$, namely,

$$m^* = \frac{\hbar^2}{(dE/dk^2)}$$  \hspace{1cm} (1.9)

In three dimensions the same formalism is carried over, but now the bands are replaced by allowed (Brillouin) surfaces and the axes of symmetry of the crystal must be taken into account.

A different approach is to start with a molecular wave function and study its behavior as the number of identical atoms is increased. In Fig. 3.5 are plotted the energy levels against interatomic distance for the $1s$ and $2s$ states of a linear array of six atoms (after Shockley).

If, then, in the limit the (almost infinite) array of the crystal is considered, the energy levels coalesce into bands. This is shown in the left-hand side of Figs. 3.6 and 3.7, where the energy bands plotted against interatomic spacing are given (after Kimball) for diamond, which is an insulator, and (after Slater) for sodium, which is a conductor.

If the lattice spacing for the particular crystal is known (as from observation), it is possible to read off from the graphs the limits of the energy bands. This is done diagrammatically on the right hand side of Figs. 3.6 and 3.7; also indicated is the position (in electron volts), of the Fermi level (as it can be calculated, for example, from Eq. 1.4 and the electron density within each band).
3. Quantum-Mechanical Systems

Fig. 3.5 Energy levels of a system of six similar atoms placed in a linear array.

From these considerations it is possible to understand the difference between conductors, insulators, and semiconductors. For diamond, for example, the valence band is completely filled (this fact follows also from the atomic structure of carbon and the deformation of the energy levels). The next available states are approximately 5.4 eV higher and hence cannot be reached by the electrons, with a consequent inhibition of their mobility; diamond therefore behaves as an insulator. For sodium, on the contrary, the Fermi level lies in the middle of an energy band, so that many states are available for the (3s) electron, which can move in the crystal freely; therefore, sodium behaves as a conductor. Pure semiconductors, such as germanium, have a configuration such that the valence band is completely filled, but the conduction band lies fairly closely to it (0.08 eV). At high enough temperatures (that is, of the order of a few thousands of degrees), the electrons in the valence band acquire enough energy to cross the gap and occupy a state in the conduction band; when this happens the material which was previously an insulator becomes intrinsically conducting.

Fig. 3.6 The energy band structure of diamond (insulator) as a function of lattice spacing. The observed lattice spacing is also indicated.

Diamond C (1s)²(2s)²(2p)²

Remaining 6 states per atom
4 states per atom
6 states per atom
Valence band (2s) (2p)²

Lattice spacing

Diagrammatic sketch

Energy

Conduction band

Valence band (2s) (2p)²

Observed lattice spacing

5.4 eV

Fig. 3.7 The energy band structure of sodium (conductor) as a function of lattice spacing. The observed lattice spacing and position of the Fermi level are also indicated.

Both the electric and thermal conductivity of a solid depend on the density and mobility of the free electrons. Completely analogous to the motion of electrons is the motion of "holes"; holes can be thought of either as "vacancies" in an almost-filled band, or as electrons with negative effective mass.† Due to their thermal energy, the carriers have a random motion characterized by \((3/2)kT = E = m^*v^2/2\). When electric potential or temperature gradients are applied, a drift velocity is superimposed on the random motion of the carriers in such a direction as to establish a steady state current flow.

2. Thermionic Emission of Electrons from Metals

It is well known that when a metal is heated to high temperatures it emits electrons, as in the case of the filament of an electron tube. To ob-

† This can be seen from Eq. 1.9 and the negative curvature of some parts of the \(E(k)\) curve of Fig. 3.4b.
serve thermionic current, both the emitter and detector are placed in an evacuated vessel and electric field acceleration is provided. However, the electric fields present and the geometrical configuration of filament and cathode complicate the interpretation of the emission process itself. We will proceed by first deriving the expression for the emitted current density (Richardson's equation) and the modifications needed when an accelerating or decelerating potential is applied; next we will give the expression for the space charge limited current† (Child's law) and consider the experimentally measurable quantities; finally, the equipment and experimental procedure will be discussed and analysis of specific data will follow.

2.1 Derivation of Richardson's Equation

Let us first consider the potential in the vicinity of a metal boundary. We assume that the potential \( \phi \) inside the metal is constant (hence \( E = 0 \)), but at the boundary proper, there must exist strong forces if the electrons are to remain contained; Fig. 3.8a shows such a square barrier potential. However, an electron which is outside a plane conducting surface is attracted to it by the force exerted between itself and its fictitious image. (The image is introduced so that for \( z \geq 0 \) the potential satisfies both Laplace's equation and the boundary conditions on the metal surface, Fig. 3.8b). Since \( F = e^2/(4 \pi \varepsilon_0 d^2) \), it follows that

\[
V = -\int E dz = -\frac{e}{16 \pi \varepsilon_0} \tag{2.1}
\]

Combining Eq. 2.1 with the square barrier of Fig. 3.8a, we obtain the more realistic image field barrier shown in Fig. 3.8c. The Coulomb potential has been cut off at \( z = z_0 \) where \( V(z_0) = -e/(16 \pi \varepsilon_0) = -w_B \); \( w_B \) is the depth of the potential well in which the free electrons are bound; or in band structure terminology, the potential energy of the bottom of the conduction band. In Fig. 3.9a the density of states inside the potential well is sketched by using Eq. 1.4; in Fig. 3.9b a plot of the density (abscissa) for various temperatures is given. Thus, free electrons with energy \( E > w_B \) can overcome the boundary and will escape if their direction of motion is towards the boundary. As is usually done in escape problems, we calculate the number of electrons within a right cylinder of unit area, and of height \( v_s \), with \( v_s > 2w_B/m \) (this gives the number of electrons that reach unit area in unit time and have the appropriate velocity component). Thus

\[
J = eN(\text{escape/sec cm}^2) = e \int_{v_s=\sqrt{(2mE/m)}}^{\infty} dN(v_s) dv_s
\]

the distribution function \( dN(v_s)/dv_s \) has been given in Eq. 1.4a, and for most values of \( T \) the exponential is very small (at \( T = 3000^\circ \text{K} \) the exponential is \( e^{-4} \)) so that

\[
\ln (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots
\]

and thus

\[
J = e^{-4\pi m kT/\hbar^2} \left[ \int_{\sqrt{(2mE/m)}}^{\infty} n_s \exp (w_B - \frac{1}{2}mv_s^2) \right] dv_s
\]

\[\text{Fig. 3.9 The energy distribution of electrons inside a metal with respect to the potential barrier. (a) The ordinate gives energy, whereas the abscissa gives the distance from the boundary; the density of horizontal lines is proportional to the number of electrons that have this particular energy. (b) The density of occupied electron states versus energy; this is shown for different Temperatures, } T.\]
3. QUANTUM-MECHANICAL SYSTEMS

We make the following change of variables

\[ u = \frac{w_F - \frac{1}{2}mv^2}{kT} \quad \text{and} \quad du = -\left(\frac{mv}{kT}\right)dv, \]

so that

\[ v_s = \left(\frac{2w_B}{m}\right)^{1/2} \to u = \frac{w_F - w_B}{kT} \]

so that

\[ J = \frac{4\pi m e k^2 T^2}{h^3} \int_{(w_F-w_B)/kT}^{-\infty} e^u du \]

giving Richardson's equation,

\[ J_0 = A_e T^3 \exp \left(-\frac{e\phi}{kT}\right) \quad (2.2) \]

with

\[ A_e = \frac{4\pi m e k^2}{h^3} = 1.2 \times 10^4 \text{ amp/m}^2\text{-deg}^2 \]

and

\[ \phi = \frac{w_B - w_F}{e} \]

is the work function in volts.

We have thus obtained Richardson's equation, which shows that the thermionic emission is dominated by an exponential temperature dependence. This dependence is so strong that the \( T^3 \) factor is completely masked and cannot be directly verified by experiment. Furthermore, the constant \( A_e \) observed experimentally is seldom in agreement with the theoretical value. This is due to the simple assumptions used in the derivation and the neglect of such effects as:

(a) Temperature dependence due to variations in \( \phi \). Because if

\[ \phi = \phi_0 + aT, \]

then

\[ J = (A_e e^{-\phi_0/kT}) T^3 \exp \left(-\frac{e\phi_0}{kT}\right) \]

where the exponential can easily alter the value of \( A_e \) by factors of 2 or larger.

(b) Quantum-mechanical reflections of the electrons at the surface. Such reflections will be equivalent to a modification of the barrier potential assumed.

(c) Nonuniformity of the emitting surface, since we really measure

\[ J = A_e T^3 \sum_i \exp \left(-\frac{e\phi_i}{kT}\right) \]

We can now consider the modifications to the emission-current density \( J_0 \) due to the application of external fields. The two cases of a retarding field and of an accelerating field are sketched in Figs. 3.10a and 3.10b, respectively.

For a retarding field, the barrier potential that the electron has to overcome is increased by the whole amount of the retarding potential \( V_0 \), so that

\[ J' = ATe^{\frac{e(\phi + V_0)}{kT}} = J_0 e^{\frac{V_0}{kT}} \quad (2.3a) \]

hence an exponential reduction of the current with increasing retarding potential.

\[ \begin{align*}
\text{(a)} & \quad \text{Retarded potential} \\
\phi &= V_0x/d \\
& \text{Towards collector} \\
\text{(b)} & \quad \text{Accelerating potential} \\
\phi &= -V_0x/d \\
& \text{Towards collector}
\end{align*} \]

Fig. 3.10 The potential seen by an electron inside a metal when an external field is applied. (a) Retarding field. (b) Accelerating field.

An accelerating field, on the other hand, as can be seen from Fig. 3.10b, can lower the potential barrier by only a small amount. For plane geometry the resultant potential is

\[ V = -\frac{e}{16\pi\varepsilon_0} \frac{V_0\varepsilon}{d} \]

which has a maximum at

\[ \varepsilon_{max} = \sqrt{\frac{ed}{16\pi\varepsilon_0 V_0}} \]
with the value

\[ V_{\text{max}} = -2 \frac{V_{\text{ap}}}{16 \pi \varepsilon_0 d}. \]

The barrier seen by the electrons is

\[ \phi = -V_{\text{max}} - w_B = -w_B + 2 \frac{V_{\text{ap}}}{16 \pi \varepsilon_0 d}. \]

and

\[ J_s = J_e \exp \left[ 0.44 (E^{11}) / T \right], \quad (2.3b) \]

where \( E \) is the electric field in volts per meter (for a plane diode \( E = V_o / d \)), and \( J_s \) is the saturation current. Because of the factor \( 0.44 / T \), the increase in emission current is very small; on a semilog plot it is proportional to the square root of the applied accelerating potential.

2.2 Child’s Law and the Measurable Quantities in the Thermionic Emission Experiment

As mentioned before, the measured values of the thermionic current are affected by space-charge effects; that is, the emitted electrons form a sheath of negative charge in the vicinity of the cathode, inhibiting further emission. It is therefore necessary to apply a positive potential to accelerate the electron toward the anode and so reduce the space-charge layer.

By solving Poisson’s equation in the space between cathode and anode and taking account of the dynamical equilibrium, we obtain expressions for the current density against voltage when accelerating fields are applied. These expressions depend on geometry and are, for a plane diode,

\[ J = \frac{4 e_0}{9} \sqrt{\frac{2 e}{m}} V_e^{9/2} d^{-2} \quad (2.4a) \]

and for a cylindrical diode

\[ J_o = \frac{4 e_0}{9} \sqrt{\frac{2 e}{m}} V_o^{9/2} r_a^{-1/2} \beta ^{-2}, \quad (2.4b) \]

where MKS units are used and \( J \) or \( J_o \) is the current density at the anode. Here \( V_o \) is the applied potential, \( d \) the anode-cathode separation for the plane diode, and \( r_a \) the anode radius for the cylindrical diode; \( \beta \) is a correction coefficient which is a function of \( (r_a / r_o) \) and can be found in the literature. Equations 2.4 are known as “Child’s law.”

As the voltage is further increased, \( J \) tends toward and finally reaches the saturation-current value given by Eq. 2.3b. We can now see what the measurable quantities are and what experimental procedure to follow.

(a) At fixed filament temperature, the anode current is measured against accelerating potential well into the saturation region. (A family of such curves is shown in a log-log plot in Fig. 3.14.) From the space-charge region we can verify Eqs. 2.4, the \( V_e^{9/2} \) dependence, and obtain a value for \( e / m \).

From the saturation region we can verify Eq. 2.3b, the \( E^{11} \) dependence of the current density, and obtain the zero field density \( J_e(T) \). It is then possible to check Eq. 2.2 (see Fig. 3.16) and find a value for the coefficient \( e \beta / k \) of the exponent (hence the work function), and for the coefficient \( A_s \).

(b) At fixed filament temperature the anode current is measured as a function of deaccelerating potential. One can verify the exponential dependence of \( J \) and determine the coefficient \( e / kT \) of the exponent (Eq. 2.3a). In principle one can again determine \( J_e(T) \), but values obtained from the saturation current are more accurate.

As is clear by now, a knowledge of the filament temperature is needed; the region of interest for pure tungsten (the cathode material used in this experiment) is in the region of 2000° K where thermionic emission currents can be reasonably detected. The measurement of the temperature cannot be made very accurately, and is based on the change in resistivity of tungsten, or by using an optical pyrometer. The special tube used in this laboratory has a small hole in the anode so that the central part of the filament can be viewed for the optical determination of the temperature.

When the change in resistivity is used, it is important to be able to separate the effects due to lead resistance and filament resistance. While the lead resistance should remain constant, nevertheless it changes, since the leads become heated by conduction from the filament. It is also important to determine correctly the room temperature resistance, since in the measuring process the filament becomes heated. In addition, the filament itself is not at uniform temperature throughout its length, but is lower by a factor of ½ at its terminal points. A correction for this effect can be included in the data.

2.3 Thermionic Emission Experiment and Results

In this laboratory a specially constructed cylindrical diode is used. The FP-400 tube manufactured by the General Electric Company is a high-vacuum tube with a pure tungsten filament. Some relevant
data† are as follows:

<table>
<thead>
<tr>
<th>Material / Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure tungsten filament, length</td>
<td>3.17 cm</td>
</tr>
<tr>
<td>diameter</td>
<td>0.013 cm</td>
</tr>
<tr>
<td>Anode, Zr coated Ni, I.D.</td>
<td>1.58 cm</td>
</tr>
<tr>
<td>Average lead resistance</td>
<td>0.08 ohms</td>
</tr>
<tr>
<td>Maximum filament voltage</td>
<td>4.75 V</td>
</tr>
<tr>
<td>Maximum filament current</td>
<td>2.5 amp</td>
</tr>
<tr>
<td>Maximum anode voltage</td>
<td>125 V</td>
</tr>
<tr>
<td>Maximum anode current</td>
<td>0.055 amp</td>
</tr>
<tr>
<td>Maximum anode dissipation</td>
<td>15 W</td>
</tr>
</tbody>
</table>

The resistivity of tungsten is $5.64 \times 10^{-6}$ ohms/cm at 27°C.

The circuit diagram is shown in Fig. 3.12. Filament heating is provided by a d-c balanced supply to minimize potential differences along the filament. The shunt and potentiometer shown in the filament circuit are used for resistivity measurements. The configuration shown in the figure is best suited for the application of accelerating potentials; for decelerating potentials a galvanometer must be introduced in the anode branch. In the latter instance voltage gradients along the filament become a serious consideration, and knowledge of the correct retarding potential is difficult to achieve unless a switching technique is used for filament heating.

When taking data, the maximum filament rating of 4.5 V should not be exceeded. Information necessary for the temperature determination is given in Fig. 3.13, which is a plot of the ratio of the resistivity of tungsten (W), obtained from independent calibration.†

Data obtained by a student‡ are shown in Fig. 3.14, which is a log-log plot of anode current against anode voltage. Since the geometry is cylindrical, we obtain the saturation current from Eq. 2.4b

$$I_s = \frac{8\pi n s}{9} \sqrt{\frac{2e}{m}} V_0^{1/2} \rho_a^{-1/2}$$


‡ D. Kohler, class of 1962.
where \( l \) is the length of the filament; for this particular tube the constant \( \alpha^{2} \) has the value 0.93. In the space-charge region, the plots are indeed straight lines and their slope is

\[
\frac{\Delta \log I}{\Delta \log V} = 1.49 \pm 0.02
\]

in agreement with the prediction of Eqs. 2.4. If we attempt, however, to obtain \( e/m \) using Eq. 2.4c, and accept as the current \( I = 0.041 \times 10^{-3} \) amp at 1 V, the result is

\[ e/m = 1.07 \times 10^{11} \text{ coul/kg} \]

while the true value is \( e/m = 1.76 \times 10^{11} \text{ coul/kg} \). This discrepancy is not surprising since only a fraction of the filament length \( l \) contributes effectively to the emission process; this is due to both thermal and collection end-effects.

To obtain the filament temperature, first the resistance of room temperature was measured and extrapolated to zero current (Fig. 3.15), with the result \( R' = 0.285 \) ohms; next the filament resistance was calculated from its geometrical dimension and found to be \( R_f = 0.135 \) ohms, so that the lead resistance must be \( R = R' - R_f = 0.15 \) ohm. From the measurements of \( R' \) it is then possible to obtain the corresponding temperature, which has been used to label the curves of Fig. 3.14.

Once the temperature is known, the zero field current \( I_0 \) can be obtained from the saturation current \( I_s \) through Eq. 2.3b; the correction is in general small. Since the geometry is cylindrical,

\[
V = \int E \cdot dr = (Er) \ln \frac{r_a}{r_f}
\]

Fig. 3.14 Data on thermionic emission. Plate current versus anode voltage as a function of filament temperature.
3. QUANTUM-MECHANICAL SYSTEMS

![Graph](image)

Fig. 3.15 "Cold" resistance of tungsten filament and leads.

hence

\[ E = \frac{V}{r} \ln \frac{r}{r_a}. \]

The results are summarized in Table 3.1, where \( J_0 = I_0/(2\pi r d) \) is the thermionic emission current density at the cathode.

### TABLE 3.1
Data Obtained from Thermionic Emission Experiment

<table>
<thead>
<tr>
<th>Filament voltage</th>
<th>3</th>
<th>3.25</th>
<th>3.5</th>
<th>3.75</th>
<th>4.0</th>
<th>4.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ) (°K)</td>
<td>2053</td>
<td>2115</td>
<td>2190</td>
<td>2260</td>
<td>2320</td>
<td>2360</td>
</tr>
<tr>
<td>( \text{mA} ) at 110 V</td>
<td>2.6</td>
<td>3.7</td>
<td>6.8</td>
<td>12.5</td>
<td>20.2</td>
<td>33.8</td>
</tr>
<tr>
<td>( J_0 ) (mA)</td>
<td>1.76</td>
<td>2.0</td>
<td>3.28</td>
<td>6.03</td>
<td>11.13</td>
<td>18.05</td>
</tr>
<tr>
<td>( J_0/T^2 ) (amp deg²-m²) X 10⁻⁴</td>
<td>0.324</td>
<td>0.582</td>
<td>0.575</td>
<td>1.59</td>
<td>2.52</td>
<td>4.18</td>
</tr>
</tbody>
</table>

A plot of the values of \( J_0/T^2 \) against \( 1/T \) in a semilog plot is given in Fig. 3.16; from it a slope

\[ \frac{e\phi}{k} = 37,500° \text{K} \]

is obtained, hence

\[ \phi = 3.22 \text{ V} \]

Similar but more precise measurements by Langmuir and Jones† yield \( \phi = 4.52 \text{ V} \), which is within the accepted value for the work function of tungsten. It is clear that values of \( A_0 \) obtained from these data depend

† From General Electric data sheet for FP-400.

in a very sensitive manner on the slope of the linear fit in Fig. 3.16. For example, using these data, we obtain \( A_0 = 33.5 \text{ amp/deg}^2\text{-m}^2 \) which is orders of magnitude off from the accepted value†

\[ A_0 = 60 \times 10^4 \text{ amp/deg}^2\text{-m}^2 \]

† This differs from the theoretical value by a factor of 2, attributed to reflections at the surface.
On the other hand, if Langmuir's value of $\phi = 4.52$ V is accepted for the slope, the data of Fig. 3.16 yield a more reasonable value:

$$A_0 = 176 \times 10^6 \text{amp/deg}^2\text{-m}^2.$$

When a retarding potential is applied, measurements with the present setup are more difficult because of the effect of voltage gradients along the filament; also space-charge effects are still quite pronounced for anode currents of the order of $10^{-4}$ amp so that only data beyond this point must be considered. For such measurements see Harriwell and Livingood, *Experimental Atomic Physics*, page 210.

### 2.4 The Stefan-Boltzmann Law

With the data obtained from the thermionic emission experiment, it is possible to verify the Stefan-Boltzmann law. This law, which can be derived on thermodynamic arguments, states that the total energy radiated per second by a black body of unit surface area is proportional to the fourth power of the temperature.

$$E = \sigma T^4 \tag{2.5}$$

where $E$ is the total power radiated from unit area, $T$ the temperature in degrees Kelvin, and $\sigma$ is Stefan's constant.

By integrating Planck's radiation law over all wavelengths, Eq. 2.5 can be obtained. Since the energy density $u$, for a frequency $v$ is

$$u, dv = \frac{8\pi k}{c^2} \frac{v^3}{\exp(hv/kT) - 1} dv \tag{2.6}$$

$$u = \int_0^\infty u, dv = \frac{8\pi k^4 T^4}{h^2 c^3} \int_0^\infty \frac{v^3}{v^4 - 1} dv = \frac{8\pi k^4}{h^2 c^3} \frac{\pi^4}{15} T^4 \tag{2.7}$$

If the total power emitted by a black body per unit area is $E$, the energy density when in equilibrium is

$$u = \frac{4}{c} E$$

so that

$$E = \frac{c}{4} \frac{8\pi k^4}{15 h^2 c^3} T^4 \tag{2.8}$$

and substitution of the constants in Eq. 2.8 gives the experimentally observed value of $\sigma = 5.64 \times 10^{-8}$ joules m$^{-2}$ deg$^{-4}$ sec$^{-1}$.

If it is then assumed that the energy loss of the filament through conduction in the leads is small, the radiated power is given

$$P = P^R f$$

---

Fig. 3.17 Verification of Stefan's law. Plot of radiated power versus filament temperature.

A plot of $P$ against $T^4$ as obtained by a student† is given in Fig. 3.17. The best fit to his data gives

$$P \propto T^{4.05}$$

† D. Owen, class of 1963.
and for Stefan's constant, $\sigma = 2.7 \times 10^{-8}$ joules m$^{-2}$ deg$^{-1}$ sec$^{-1}$ which is of the correct order of magnitude, and smaller than $\sigma$.

In concluding, we note that our results have confirmed the exponential dependence on temperature of the thermionic current (Richardson's equation), and the phenomena of space charge. Also, qualitative agreement has been achieved with the accepted values of the parameters involved; similarly, agreement has been achieved for Stefan’s law.

3. Some Properties of Semiconductors

3.1 GENERAL

We have seen in the first section how a free-electron gas behaves, and what can be expected for the band structure of a crystalline solid. In the second section we applied the principle of free-electron gas behavior to the emission of electrons from metals, and in the present section we will apply both principles to the study of some properties of semiconductors which can be verified easily in the laboratory.

As mentioned before, a semiconductor is a crystalline solid in which the conduction band lies close to the valence band, but is not populated at low temperatures; semiconductors are unlike most metals in that both electrons and holes are responsible for the properties of the semiconductor. If the semiconductor is a pure crystal, the number of holes (positive carriers, $p$) is equal to the number of free electrons (negative carriers, $n$), since for each electron raised to the conduction band, a hole is created in the valence band; these are called the intrinsic carriers.

All practically important semiconductor materials, however, have in them a certain amount of impurities which are capable either of donating electrons to the conduction band (making an $n$-type crystal) or of accepting electrons from the valence band, thus creating holes in it (making a $p$-type crystal). These impurities are called extrinsic carriers; in such crystals $n \neq p$.

Let us then first look at the energy-band picture of a semiconductor as it is shown in Fig. 3.18; the impurities are all concentrated at a single energy level, normally lying close to the conduction band, the density of states at this energy level is given by $D(E) = \text{constant}$ times density of states) and setting it equal to the electron density. It is clear, however, that if we are to have as many empty states in the valence band as occupied ones in the conduction band, the Fermi level must lie exactly in the middle of the forbidden gap (because of the symmetry of the trailing edge of the distribution). In Fig. 3.18, the density of states is shown to the right and the Fermi distribution function to the left. We measure the position of the Fermi level from the conduction band and define it by $E_F$; the exact value of $E_F$ is

$$E_F = -\frac{E_g}{2} + kT \ln \left( \frac{m^*_e}{m^*_h} \right)^{1/2}$$

(3.1)

Since the Fermi level lies below the conduction band, $E_F$ is a negative quantity; $E_g$ is the energy gap always taken positive and $m^*_e$ and $m^*_h$ are the effective masses of holes and electrons, respectively. If $w_c$ and $w_p$ stand for the actual position of the conduction band and Fermi level above the zero point energy, then

$$w_p = w_c + E_F$$

To find the number of electrons in the conduction band (or holes in the valence band) we simply substitute Eq. 3.1 for $w_p$ into Eq. 3.1, multiply by the density of states, and integrate over $w$ from $w = w_c$ to $+ \infty$. When, however, the exponent

$$-(w_p - w) \approx \frac{E_g}{2} + E \gg kT$$

(3.2)

the Fermi distribution degenerates to a Boltzmann distribution. (Here $E$ is the energy of the electrons as measured from the top of the conduction band; obviously it can take either positive or negative values.) With this

† Note that the tungsten filament is not a perfect black body; the emissivity of a hot tungsten filament is usually taken as one third.

† If the effective masses of $p$- and $n$-type carriers are the same.
assumption the integration is easy, yielding
\[ n = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{\frac{E_d}{kT}} \approx \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\frac{E_d}{kT}} \] (3.3a)

similarly,
\[ p = \left(\frac{2\pi m_h kT}{h^2}\right)^{3/2} e^{-\left(E_a + E_F\right)/kT} \approx \left(\frac{2\pi m_h kT}{h^2}\right)^{3/2} e^{-\frac{E_a}{kT}} \] (3.3b)

It is interesting that the product \( np \) is independent of the position of the Fermi level—especially if we take \( m_e = m_h \)
\[ n_e^2 = np = 2.31 \times 10^{23} \text{ cm}^{-3} \text{ e}^{-\frac{E_d}{kT}} \]

Thus it should be expected that as the temperature is raised, the intrinsic carriers of a semiconductor will increase at an exponential rate characterized by \( E_d/2kT \). This temperature is usually very high since \( E_d \approx 0.7 \text{ V} \) (see Eqs. 3.5).

We have already mentioned that impurities determine the properties of a semiconductor, especially at low temperatures where very few intrinsic carriers are populating the conduction band. These impurities, when in their ground state, are usually concentrated in a single energy level lying very close to the conduction band (if they are donor impurities) or very close to the valence band (if they are acceptors). As for the intrinsic carriers, the Fermi level for the impurity carriers lies halfway between the conduction (valence) band and the impurity level; this situation is shown in Figs. 3.19a and 3.19b. If we make again the low temperature approximation of Eq. 3.2, the number of electrons in the conduction band is given by
\[ n = N_d \left(\frac{2\pi m_k T}{h^2}\right)^{3/2} e^{-\frac{E_d}{kT}} \] (3.4)

where \( N_d \) is the number of donors and \( E_d \) the separation of the donor energy level from the conduction band. In writing Eq. 3.4, however, care must be exercised because the conditions of Eq. 3.2 are valid only for very low temperatures. Note, for example, that for germanium
\[ E_x = 0.7 \text{ eV}, \quad \text{and for } kT = 0.7 \text{ eV}, \quad T = 8,000^\circ \text{ K} \]
whereas
\[ E_d = 0.01 \text{ eV}, \quad \text{and for } kT = 0.01 \text{ eV}, \quad T = 120^\circ \text{ K} \] (3.5)

Thus at temperatures \( T \gtrsim 120^\circ \text{ K} \) most of the donor impurities will be

\[ \text{Fig. 3.19} \quad \text{Same as Fig. 3.18 but with the addition of impurities. (a) The impurities are of the donor type and lie at an energy slightly below the conduction band. (b) The impurities are of the acceptor type and lie slightly above the valence band. Note the shift of the Fermi level as indicated by the dotted line.} \]

in the conduction band and instead of Eq. 3.4 we will have \( n \approx N_d \); namely, the number of impurity carriers becomes saturated. Once saturation has been reached the impurity carriers in the conduction band behave like the free electrons of a metal.

In the experiments to be discussed below, the lowest temperature achieved was \( T \approx 84^\circ \text{ K} \); for germanium this still corresponds to almost complete saturation of the impurity carriers. We will, however, be able to study the gradual increase, as a function of the temperature, of the intrinsic carriers of germanium.

### 3.2 Resistivity

We know already that conduction in solids is due to the motion of the charge carriers under the influence of an applied field. We define the following symbols:

- **\( J \)** = current density
- **\( \sigma \)** = conductivity, such that \( \sigma E = J; \rho = 1/\sigma = \text{resistivity} \)
- **\( \mathbf{E} \)** = electric field
- **\( \mathbf{v} \)** = drift velocity
- **\( \mathbf{E} \)** = electric field
- **\( \mu \)** = mobility; such that \( \mu \mathbf{E} = \mathbf{v} \)
- **\( m^* \)** = effective mass; \( m_e^*, m_h^* \) for electrons, holes
- **\( \lambda \)** = mean free path between collisions
From simple reasoning, the current density is
\[ J = \frac{I}{A} = e(n\bar{v}_a - p\bar{v}_b) \] (3.6)
which is equivalent to the carrier density multiplied by the drift velocity.
Thus
\[ s = \frac{1}{2} \alpha t^2 = \frac{1}{2} \frac{e|E|}{m^*} t^2 \]
and
\[ \bar{v} = \frac{s}{t} = \frac{1}{2} \frac{e|E|}{m^*} t \]
in terms of the mean free path, \( t = \lambda/v \), where \( v \) is now the thermal velocity,
\[ \frac{m^* \lambda^2}{2} = \frac{3}{2} kT \]
thus
\[ \bar{v} = \frac{eE}{2\sqrt{3kTm^*}} \]

If only one type of carrier is present,
\[ J = en\bar{v} = en\mu E \]
and
\[ \sigma = en\mu = \frac{ne\lambda}{2\sqrt{3kTm^*}} \]
Thus, if (a) the number of carriers is constant, and (b) the mean free path remains constant, the conductivity should decrease as \( T^{-1/2} \). The first of these conditions holds in the extrinsic region after all impurity carriers are in the conduction band; the mean free path, however, is not constant, because higher temperatures increase lattice vibrations, which in turn affect the scattering of the carriers. A simple calculation suggests a \( 1/kT \) dependence for \( \lambda \), so that
\[ \sigma = ne\mu = C \frac{ne^2}{m^*} T^{-1/2} \] (3.7)
where \( C \) is a constant. We will see that this dependence is not always observed experimentally.

### 3.3 The Hall Effect

It is clear from Eq. 3.6 that conductivity measurements cannot reveal whether one or both types of carriers are present, nor distinguish between them. However, this information can be obtained from Hall effect measurements, which are a basic tool for the determination of mobilities. The effect was discovered by E. H. Hall in 1879.

![Fig. 3.20 Schematic arrangement for the measurement of the Hall effect of a crystal.](image)

Consider a simple crystal mounted as in the Fig. 3.20, with a magnetic field \( H \) in the \( z \) direction perpendicular to contacts 1, 2 and 3, 4. If current is flowing through the crystal in the \( x \) direction (by application of a voltage \( V_x \) between contacts 1 and 2), a voltage will appear across contacts 3, 4. It is easy to calculate this (Hall) voltage if it is assumed that all carriers have the same drift velocity. We will do this in two steps: (a) by assuming that carriers of only one type are present, and (b) by assuming that carriers of both types are present.

(a) One type of carrier. The magnetic force on the carriers is \( \mathbf{F}_m = e(\mathbf{v} \times \mathbf{H}) \) and it is compensated by the Hall field \( \mathbf{F}_H = eE_H = eEnH \) thus \( tH = E_H \), but \( \bar{v} = \mu E_x \), hence \( E_H = H\mu E_x \). The Hall coefficient \( R_H \) is defined as
\[ |R_H| = \frac{E_y}{J_x H} = \frac{\mu E_x}{J_x} = \frac{\mu}{\sigma} = \frac{1}{ne} \] (3.8a)
Hence for fixed magnetic field and fixed input current, the Hall voltage is proportional to \( 1/n \). It follows that
\[ \mu_H = R_H \sigma, \] (3.8b)
providing an experimental measurement of the mobility; \( \mu_H \) is expressed in \( \text{cm}^2 \text{coulomb}^{-1} \), and \( \sigma \) in \( \text{ohm}^{-1} \text{cm}^{-1} \); thus \( \mu \) is expressed in units of \( \text{cm}^2 \text{volt}^{-1} \text{sec}^{-1} \).

In most experiments the voltage across the input is kept constant, so that it is convenient to define the Hall angle as the ratio of applied and measured voltages:
\[ \phi = \frac{V_x}{E_x} = \frac{E_x}{E_{xH}} = \mu \frac{t}{l} H \] (3.8c)
where \( t \) is the length and \( t \) the thickness of the crystal.
The Hall angle is proportional to the mobility, and

\[ \rho V_y = \left( \frac{l}{H} \right) \frac{1}{n e} \]  

(3.9)

is again proportional to \(1/n\) and thus to \(|R_H|\).

(b) Two types of carriers. Now it is important to recognize that for the same electric field \(E_x\), the Hall voltage for \(p\) carriers will have opposite sign from that for \(n\) carriers. (That is, the Hall coefficient \(R\) has a different sign.) Thus, the Hall field \(E_y\) will not be able to compensate for the magnetic force on both types of carriers and there will be a transverse motion of carriers; however, the net transverse transfer of charge will remain zero since there is no current through the 3, 4 contacts; this statement is expressed as

\[ e(v_x^n - v_y^n) = 0 \]

while

\[ e(v_x^p - v_y^n) = J_x \]

and

\[ e(\mu_x^p + \mu_y^n) = \sigma \]

where the mobility is always a positive number; however, \(v_x^n\) has the opposite sign from \(v_x^p\), but

\[ v_y = \frac{g}{l} = \left( \frac{1}{2m^*} \right) \frac{1}{l} \]

where

\[ F^+ = e[(v_x^p \times H) - E_x] \]

\[ F^- = -e[(v_x^n \times H) - E_x] \]

Thus

\[ v_x^p = \frac{1}{2m^*} \frac{e}{\mu_x^p} \left[ \mu_x^p E_x - E_y \right] = \mu_x^p (\mu_x^p E_x - E_y) \]

\[ v_x^n = \frac{1}{2m^*} \frac{e}{\mu_x^n} \left[ \mu_x^n E_x - E_y \right] = \mu_x^n (\mu_x^n E_x - E_y) \]

and thus

\[ \mu_x^p (\mu_x^p E_x - E_y) - \mu_x^n (\mu_x^n E_x + E_y) = 0 \]

\[ E_y = E_x H \frac{(\mu_x^p \mu_x^n - \mu_x^n \mu_x^n)}{\mu_x^p + \mu_x^n} \]

and for the Hall coefficient \(R_H\)

\[ R_H = \frac{E_y}{J_x H} = \frac{E_y}{\sigma E_x H} = \frac{\mu_x^p \mu_x^n - \mu_x^n \mu_x^n}{\sigma (\mu_x^p + \mu_x^n)^2} \]  

(3.10)

Equation 3.10 correctly reduces to Eq. 3.8 when only one type of carrier is present.†

Since the mobilities \(\mu_x^p\) and \(\mu_x^n\) are not constants but functions of \(T\), the Hall coefficient given by Eq. 3.10 is a function of \(T\) and it may become zero and even change sign. In general, \(\mu_x^p > \mu_x^n\) so that inversion may happen only if \(p > n\); thus "Hall coefficient inversion" is characteristic only of "p-type" semiconductors.

At the point of zero Hall coefficient, it is possible to determine the ratio of mobilities \(b = \mu_x^p/\mu_x^n\) in a simple manner. Since \(R_H = 0\), we have from Eq. 3.10

\[ nb^2 - p = 0 \]  

(3.11)

Let \(N_a\) be the number of impurity carriers for this "p-type" material; then in the extrinsic region

\[ p = N_a \quad n = 0 \]

whereas in the intrinsic region

\[ p = N_a + N \quad n = N \]

and Eq. 3.11 becomes

\[ n = \frac{N_a}{b^2 - 1} \]  

(3.12)

We can also express the conductivity \(\sigma\) at the inversion point, \(T = T_0\), in terms of the mobilities

\[ \sigma_0 = e[(\mu_x^p + \mu_x^n)(N_a + n)] \]  

(3.13)

and this value can be directly measured. Further, by extrapolating conductivity values from the extrinsic region to the point \(T = T_0\) we obtain

\[ \sigma(T = T_0) = \sigma_0 N_a(T = T_0) \]

It therefore follows that

\[ \frac{\sigma_0}{\sigma(T = T_0)} = \frac{N_a + n(1 + b)}{N_a} \]  

(3.14)

† Both Eq. 3.8 and Eq. 3.10 have been derived on the assumption that all carriers have the same velocity; this is not true, but the exact calculation modifies the results obtained here by a factor of only \(3\pi/8\).
agree with directly measured values, and a distinction between the two is made; this will become clear when the experimental data are analyzed (Eq. 3.18 and Eq. 3.19).

3.4 Experimental Arrangement and Procedure

The sample, a small crystal of germanium, is mounted in a cryostat; a drawing of the assembly is shown in Fig. 3.21. The small dewar placed on the top permits the use of a magnet with only a 2-in. poleface separation. The heat is drawn from the sample chamber through the copper-brass rod into the liquid nitrogen heat sink. This allows the sample chamber to reach approximately 80° K. The liquid nitrogen must be kept up to level throughout the entire experiment. To raise the temperature of the sample chamber, a heating coil is placed just below it. The coil is wound non-inductively of 9 ft of No. 32 cotton-covered resistance wire; between each layer of the winding, metal foil is placed to conduct the heat quickly to the copper chamber. The maximum current is 1.5 amp and the heating coil should not be operated without liquid nitrogen in the large dewar (the maximum current with no liquid nitrogen is 35 ma.). At a current of about 1.3 amp the chamber will be at room temperature.

To measure the temperature, a copper-constantan thermocouple is fastened to the outside of the sample chamber. The standard junction is in ice-water at 0°C and Fig. 3.22 gives the appropriate calibration on this basis.
The equilibrium time is of the order of 15 min; however, it is not necessary to wait this long between points when taking data. Measurements should start at the lowest temperature, and during the experiment the heating coil current should be kept slightly in advance of equilibrium so as to maintain a slow but steady rise in temperature; the average millivolt reading should be recorded.

One method of mounting the crystal and making the contacts (used in this laboratory) is shown in Fig. 3.23. The copper stub used for one of the sample current contacts also provides a low-resistance heat path to keep the germanium at the sample chamber's temperature. The two wires soldered on each side of the germanium crystal allow the measurement of the Hall voltage at the top or bottom of the sample and the measurement of the conductivity on either side of the sample. Use of fine wires provides isolation from room temperature. The finite extent of the side contacts does not, to first order, affect the conductivity measurements.

![Fig. 3.23 Mounting and electrical connections to a crystal sample.](image)

There are two problems in making the solder contacts. One is wetting the germanium with solder (that is, making the solder contact stick) and the other is to avoid making a "rectifying junction." To wet the germanium it is necessary first to etch it for about 30 sec in a solution of three parts hydrofluoric acid, three parts glacial acetic acid, and four parts nitric acid; this is called the CP4 etching solution. To avoid making a rectifying junction, the surface of the germanium where the contact is to be made must be destroyed; this is best done with a small Swiss file. A colloidal mixture of acid flux and solder is then used (it looks like a gray paste and is called "plumber's solder"). A very quick etch after the contacts have been made helps to remove any flux which would change the conductivity measurements. After etching, the germanium should be handled only with clean tweezers.

A schematic diagram of the measuring circuits is shown in Fig. 3.24. Resistivity is usually measured across contacts 1, 2 and an a-c bridge should be used; this is a more accurate measurement and also reduces the effects of rectifying contacts. There are provisions to reverse the sample current, to measure the Hall voltage at either the top or bottom of the sample, and to balance out the zero field potential. The zero field potential comes from two sources: the first is due to the fact that the Hall contacts are not quite opposite each other; since there is a potential gradient due to the sample current, a part of this gradient will be seen between the two contacts. The second source is from the contact potential and the "rectifying action" of the contacts. To measure the Hall voltage, a Keithley electrometer is used, since its high input impedance does not affect the Hall voltage; the sample current is provided from a d-c battery, hence at fixed voltage. The Hall voltage must be measured for both directions of the magnetic field and should always be properly zeroed when the field is off; before the voltage is measured, the crystal must be rotated in the field until the position of maximum voltage is reached.

An experienced experimenter can take all the necessary data in one run. The sample is usually cooled to liquid nitrogen and then the temperature is slowly raised by control of the heater current. The following data should
be recorded for every temperature point:
(a) Thermocouple reading
(b) Resistivity; magnetic field off
(c) Resistivity; magnetic field on
(d) Hall voltage, with field forward, off, reversed
(e) Thermocouple reading

Less experienced persons are better advised first to make a measurement of the resistivity only, over the whole temperature range from 80° K to 330° K, and then to measure the Hall voltage separately; as usual, it is advisable to plot the data as it is obtained so as to know where a greater density of measurements is desirable.

3.5 Analysis of Data

Data on Hall effect and resistivity, obtained by students† using a low impurity germanium crystal, are presented and analyzed below.

Table 3.2 gives the raw data; the dimensions of the crystal were $l = 1.09$, $t = 0.183$, $w = 0.143$ cm (see Fig. 3.20). A fixed voltage of 1.32 V was applied across the long end of the sample, and the magnetic field was $(500 \pm 40)$ gauss. The Hall voltage was measured across the $t$ dimension of the crystal.

From the data of the Table 3.2 the following plots were made:

(a) \( \rho = 1/\sigma = RA/l \) hence‡ \( \rho = R \times 2.42 \times 10^{-2} \) ohms-cm; this is shown in a semilog plot against $1/T$ in Fig. 3.25. We note that for $T < 290°$ K, conduction is due mainly to the impurity carriers; this is the extrinsic region. For $T > 280°$ K, electrons are transferred copiously from the valence band into the conduction band and the crystal is in the intrinsic region. From the slope of the intrinsic region and making use of Eq. 3.3b, we have \( \rho = 1/n = \exp((E_g/2kT)) \) and thus \( n = E_g/2kT \). Hence

\[
E_g = \frac{1}{2k} \frac{\Delta \log \rho}{\Delta (1/T)} = \frac{1.81 \times 10^4}{0.4343} \quad (3.16)
\]

which leads to $E_g = 0.72 \pm 0.07$ eV, in agreement with the accepted value.

(b) A log-log plot of the resistivity in the extrinsic region against $1/T$ shown in Fig. 3.26. If a power law as in Eq. 3.7 is applicable, we would have \( \rho \propto (1/T)^\alpha \) and hence

\[
\alpha = \frac{\Delta \log \rho}{\Delta \log (1/T)} = -2.0 \pm 0.1 \quad (3.17)
\]

† E. Yudowsky and P. Schreiber, class of 1962.
‡ Note that here $R$ is the resistance of the sample and not the Hall coefficient $R_H$.
Since in this region the carrier density is constant, this gives for the mobility a dependence

$$\mu = CT^{-1/2}$$  \hspace{1cm} (3.18)

which is in disagreement with the prediction of Eq. 3.7. It is, however, the correct value for germanium, indicating that the simplified calculations used in deriving Eq. 3.7 are not completely adequate.

Turning now to the Hall-voltage measurements of Table 3.2, we can form the quantities defined by Eq. 3.8c.

$$\frac{V_H}{V_S} = \phi = \mu H \frac{l}{l}$$

hence

$$R_H = \phi R \frac{v}{H}$$

---

**Fig. 3.25** The resistivity of a pure germanium crystal as a function of inverse temperature. For $T < 200^\circ$ K, conduction is due mainly to the impurity carriers (extrinsic region); for $T > 200^\circ$ K, conduction is due to electrons transferred to the conduction band (and the corresponding holes created in the valence band): this is the intrinsic region.

**Fig. 3.26** A log-log plot of the resistivity of germanium in the extrinsic region versus $1/T$. It is assumed that the number of carriers is independent of $T$ since saturation of the impurity carriers has already been reached.
and the Hall mobility

\[
\mu_H = R_{H2} = \phi \left( \frac{e}{t} \right) H^{-1}
\]

The Hall mobility so obtained is shown in a log-log plot against \( T \) in Fig. 3.27. Since the Hall coefficient changes sign, we can immediately recognize that the crystal is of the \( p \) type; the inversion temperature of this particular sample is found to be

\[
T_s = (323 \pm 3) \, ^\circ \text{K}
\]

From the slope of the \( \mu_H \) curve in the extrinsic region in Fig. 3.27, we obtain

\[
\mu_H = C(T - T_s)^{-1/2}
\]

which is different from Eq. 3.18 and is in agreement with conclusions of other observers; this is the reason why a distinction between the Hall mobility \( \mu_H \) and the drift mobilities \( \mu_D \) obtained from resistivity measurements is made.

By extrapolating Eq. 3.19 to the inversion temperature, we obtain the hole mobility \( \dagger \) at \( T = T_s = 323^\circ \)

\[
\mu_H(h) = 2.7 \times 10^4 \, \text{cm}^2/\text{V-sec.}
\]

We can now apply the analysis indicated in Section 3.4, which led to Eq. 3.11. From Fig. 3.25 we have \( R_e(T - T_s) = 2150 \, \text{ohms and } R_s = 500 \, \text{ohms leading to } b = 1.31 \pm 0.2 \). Thus we obtain for the electron mobility at \( T = T_s = 323^\circ \)

\[
\mu_H(e) = 3.5 \times 10^3 \, \text{cm}^2/\text{V-sec,}
\]

both results being in agreement with the accepted values.

From the Hall coefficient in the extrinsic region, we can also obtain an order of magnitude for the density of impurity carriers. Since in that region only one type of carrier is present, \( ne = 1/R_H \), and since

\[
R_H = \frac{\phi R_w}{H} \simeq 8 \times 10^8 \, \text{cm}^2/\text{coulomb}
\]

which is reasonable for this sample, indicating an impurity concentration of the order of two parts in \( 10^6 \).

From the data of Table 3.2 it can be further noticed that the resistance of the sample changes when the magnetic field is turned on. This phenomenon, called magnetoresistance, is due to the fact that the drift velocity of all carriers is not the same. With the magnetic field on, the Hall voltage \( V = E_B = | v \times H | \) compensates exactly the Lorentz force for carriers with the average velocity; slower carriers will be overcompensated, and faster ones undercompensated, resulting in trajectories that are not along the applied external field. This results in an effective decrease of the mean free path and hence an increase in resistivity.

\[\dagger\] Note that 1 gauss = \( 10^{-4} \) weber/m\(^2\) = \( 10^{-4} \) V-sec/cm\(^2\).
4. Sketch of \(p-n\) Semiconductor Junction Theory

As mentioned before, semiconductor materials with high impurity concentration, when properly combined, form a transistor. Transistors (of the type most used today) consist of two junctions of dissimilar-type semiconductors, one \(p\) type and one \(n\) type; the intermediate region, the base, is usually made very thin. We will briefly sketch the behavior of such a \(p-n\) junction and then see how the combination of two junctions can provide power amplification; for this we will use our knowledge of the band structure of semiconductors and the position of the Fermi level, as developed previously (Figs. 3.18 and 3.19).

When two materials with dissimilar band structure are joined, it is important to know at what relative energy level one band diagram lies with respect to the other: the answer is that the Fermi levels of both materials must be at the same energy position when no external fields are applied; this is shown in Fig. 3.29.

From the energy diagram of the figure, it follows that only electrons with \(E_e > \Delta W_n\) will be able to cross the junction from the \(n\) material into the \(p\) region and only holes with \(E_h > \Delta W_p\) from the \(p\) region to the \(n\) region. Holes in the \(n\) region or electrons in the \(p\) region are called "minority carriers." Indeed, there will be diffusion of some minority carriers across the junction, but since no electric field is present these carriers will remain in the vicinity of the junction.

If now a reverse bias is applied—that is, one that opposes the further motion of the minority carriers—the Fermi levels will become displaced by the amount of the bias, as shown in Fig. 3.30a. We see that the barriers

\[\text{Fig. 3.29} \quad \text{Structure of the energy bands at the junction of an} \quad n\text{-type and a} \quad p\text{-type semiconductor.}\]
\( \Delta W_n \) and \( \Delta W_p \) are increased by almost the full voltage, making any motion of minority carriers across the junction very improbable. Fig. 3.30b, on the other hand, shows the situation when forward bias is applied (favoring the motion of minority carriers). The Fermi levels are now displaced in the opposite direction so that the barriers are lowered. However, the full bias voltage does not appear as a difference between the Fermi levels because dynamic equilibrium prevails. There is a continuous flow of minority carriers in the direction of the electric field (holes obviously moving in the opposite direction from electrons) and as a result a potential gradient exists along the material; thus the entire bias voltage does not necessarily appear at the junction itself.

We will now consider two junctions put together; in Fig. 3.31a, p-type, n-type, and again p-type material are joined. When no bias is applied, we expect the Fermi levels to be at the same position, with the resulting configuration shown in the diagram; in agreement with our previous conclusions from the consideration of a simple junction, we see that barriers exist for the motion of holes from the p regions into the n region, and also for the motion of electrons from the n region into either of the p regions.

Figure 3.31b shows the double junction under operating biases; note that one junction is biased forward, the other is biased in the reverse direction. The n-type material common to both junctions is called the base, while the p type of the forward-biased junction is the emitter; the p-type material of the reverse junction is the collector. A completely symmetric device consisting of n-p-n materials will perform similarly when the biases are reversed. From the energy diagram of Fig. 3.31b we can see that by varying the emitter junction bias we can control the injection of minority carriers into the base region; if the base region is made thin, it is possible for these holes to reach the collector junction, at which point they will immediately cross it, since it represents a gain in potential energy. If \( h_B \) is the minority carrier current injected into the base over a potential barrier \( \Delta W_B(EB) \), the power required for injection is \( P_{in} = h_B \Delta W_B(EB) \); similarly if \( h_C \) is the hole current into the collector down a potential drop \( \Delta W_C(BC) \), the power gained is \( P_{out} = h_C \Delta W_C(EC) \). Thus if \( P_{out} > P_{in} \), the device is a power amplifier; since usually \( \Delta W(EB) \gg \Delta W(EB) \), it suffices for \( h_C \approx h_B \) to give power gain.

Below we give, without proving, some quantitative formulas for the gain factors of a junction transistor. A detailed discussion can be found in Dunlap, "An Introduction to Semiconductors."

We introduce the following symbols and definitions:

- \( h_B, e_B \) hole current out of emitter, electron current into emitter
- \( h_C, e_C \) hole current, electron current into collector
- \( I_E, I_B, I_C \) emitter, base, collector current where hole current leaving a region is designated as positive
- \( w \) width of base region
- \( D_n, D_p \) diffusion coefficients for electrons, holes in base region
- \( L_E, L_B \) diffusion lengths in emitter, base
- \( \tau_n, \tau_p \) lifetime of minority carriers in emitter, base
- \( n_E, p_B \) concentration of minority carriers (electrons) in emitter, (holes) in base
- \( \omega \) angular frequency
3. QUANTUM-MECHANICAL SYSTEMS

i. \sqrt{-1}

\beta \quad h_\beta/h_\gamma \text{ diminution factor}

\gamma \quad h_\beta/I_B \text{ injection efficiency}

\alpha \quad I_C/I_B \text{ current gain}

If \( h_B \) holes are injected from the emitter into the base, because of recombination, only \( h_C \) reach the collector junction. Thus:

\[ h_C = h_\beta \\beta \]

where \( 0 < \beta < 1 \), and \( \beta \) is given by \( \beta = \text{sech}(w/L_B) \). Further

\[ I_B = h_B + e_B = h_B \left( 1 + \frac{e_B}{h_B} \right) = \frac{h_B}{\gamma} \]

where \( 0 < \gamma < 1 \), and is given by

\[ \frac{1}{\gamma} = 1 + \frac{e_B}{h_B} = 1 + \frac{D_n \cdot n_B L_B}{D_p \cdot p_B L_B} \tanh \left( \frac{w}{L_B} \right) \]

Also,

\[ I_C = -h_C + e_C \quad I_B = -e_C - e_B + (h_C - h_B) \]

with the obvious conservation expression

\[ I_C = -(I_B + I_B) \]

Since the collector leakage current \( e_C \) is very small, we can neglect it and obtain

\[ \alpha = \frac{I_C}{I_B} \approx \frac{h_C}{h_B} = \frac{h_\beta}{h_\gamma} = \beta \gamma \]

Usually \( w/L_B \ll 1 \) and then \( \gamma \approx 1 \), so we obtain the further approximate expressions

\[ \alpha \approx \beta = \text{sech} \left( \frac{w}{L_B} \right) \approx \left[ 1 + 2 \left( \frac{w}{L_B} \right)^2 \right]^{-1} \]

indicating the importance of a thin base region if current gains close to unity are to be achieved. For a time-varying signal with angular frequency \( \omega \), the above expression is modified to

\[ \alpha \approx \beta \approx \left[ 1 + \frac{1}{2} (1 + i \omega \tau_B) \left( \frac{w}{L_B} \right)^2 \right]^{-1} \]

indicating phase shifts and reduction of gain at frequencies of the order of the reciprocal lifetime of the minority carriers in the base.

REFERENCES

For the material covered in Sections 3, 4, and 5 the reader may also consult the following texts:


5. Contact and Thermoelectric Effects at Junctions of Metals

We now turn our attention again to metals; as mentioned in Sections 1 and 2, the free electrons can be thought of as being all in the continuum of the Fermi sea, with the density of states proportional to \( \sqrt{E} \) and not restricted to allowed energy bands. If two metals with different Fermi energies \( w_F(A) \) and \( w_F(B) \) and different work functions \( \phi(A) \) and \( \phi(B) \) are joined, the energy diagram will be as shown in Fig. 3.32. Again equilibrium requirements impose the condition that the Fermi levels be at the same potential.

Let us consider, then, an electron in metal \( B \) that just overcomes the work function \( \phi_B \) and is emitted from the metal surface. If while outside the metal the electron moves towards the dissimilar metal \( A \) (which is joined to \( B \)), it clearly sees a retarding potential \( \phi_A - \phi_B \) until it enters metal \( B \). This is called the contact potential; note that we had to correct for it on several occasions in discussions in Chapter 1. For an electron emitted from \( A \) and traveling towards \( B \), the contact potential is an accelerating one \( \phi_B - \phi_A \). Thus we see that contact potential differences (epd) arise when current flow through dissimilar metals is completed with a section in free space.

Next we will briefly mention three interrelated phenomena connecting the reversible flow of heat with that of current in a metal and vice versa. An application of these effects is the thermocouple of which we have repeatedly made use.

(a) The Peltier effect (1834): Let a circuit be completed through two dissimilar metals and a current flow through the junction. Reversible (not Joule) heating or cooling of the junction then occurs depending on the
direction of current flow. Analytically we express the effect as follows

\[ Q = \tau_{AB}(T)q \]

(5.1)

where \( Q \) is the amount of heat liberated, \( q \) is the total charge that crosses the junction, and \( \tau_{AB}(T) \) is the Peltier coefficient which depends on the temperature.

(b) The Thomson effect (1856): This is complementary to the Peltier effect and is the reversible heating or cooling of a conductor through which current flows and along which a temperature gradient exists. Consider a conductor \( AB \) with no current flowing but with its two end points at a different temperature \( T_A > T_B \); there will be heat transfer across it, effected through the motion of high-energy electrons from \( A \) towards \( B \); to keep the charge transfer equal to zero, low-energy electrons will move from \( B \) towards \( A \); this is shown in Fig. 3.33a. The following must hold

\[ j = e(n_{2}v_{2} - n_{1}v_{1}) = 0 \]

(5.2a)

and if \( S \) is the rate of heat transfer and \( E_z = E_A - E_B \), \( E_1 = E - E_B \) are the excess or deficit from the mean thermal energy,

\[ \frac{\Delta Q}{\Delta t} = (n_{2}v_{2}E_2 + n_{1}v_{1}E_1) = S \]

(5.3a)

If now a net current flows to the right (Fig. 3.33b),

\[ j' = e(n_{2}'v_{2}' - n_{1}'v_{1}') > 0 \]

(5.2b)

and

\[ \frac{\Delta Q}{\Delta t} = n_{2}'v_{2}'E_2 + n_{1}'v_{1}'E_1 = S' \neq S \]

(5.3b)

Namely, a change of \( \Delta Q/\Delta t \) occurs; the balance of the heat is being supplied

\[
\begin{align*}
\text{If} & \quad I_A > I_B & \quad \text{If} & \quad I_A > I_B \\
& \quad n_2v_2 & \quad n_1v_1 \\
& \quad n_2'v_2' & \quad n_1'v_1' \\
\end{align*}
\]

Fig. 3.33 A conductor along which a temperature gradient exists. (a) No net current flowing. (b) Current flowing to the right.

or absorbed by the lattice. The situation is reversed when the current flows to the left.

Analytically, we express the Thomson effect as follows:

\[ \frac{dQ}{dx} = \sigma_A q \left( \frac{dT}{dz} \right) \]

(5.4)

where \( dQ/dx \) is the heat absorbed (or liberated) per unit length of the metal; \( q \) is the total electric charge that has flowed through, and \( dT/dz \) the temperature gradient; finally \( \sigma_A \) is the Thomson coefficient and it depends on the metal. Conversely if a temperature gradient exists in a metal, an electromotive force (emf) will appear at its ends. Since the product of the charge and of the electromotive force must equal the total work done, we find by integrating Eq. 5.4

\[ \int_A^B dQ = q \int_{T_A}^{T_B} \sigma_A dT \]

hence

\[ \text{emf (Thomson)} = \int_{T_A}^{T_B} \sigma_A dT \]

(5.5)

(c) The Seebeck effect (1822): This is the appearance of an electromotive force in a circuit made of two dissimilar metals when their junctions are held at different temperatures—namely, a thermocouple. It has to be a combination of the two previously discussed effects and is shown in Fig. 3.34. The two metals will have dissimilar Thomson coefficients \( \sigma_A \) and \( \sigma_B \), and the junctions (at different temperature) will have dissimilar Peltier coefficients; the electromotive force is usually measured with
a potentiometer—namely, at zero current. Combining Eqs. 5.1 and 5.5 we obtain

\[
\text{emf} = \pi_{AB}(T_2) - \pi_{AB}(T_1) + \int_{T_1}^{T_2} \sigma_A dT - \int_{T_1}^{T_2} \sigma_B dT \quad (5.6)
\]

Taking the derivative with respect to the temperature, we obtain the \textit{thermoelectric power}, \( E_{AB} \) of two dissimilar metals

\[
E_{AB} = \frac{d(\text{emf})}{dT} = \frac{d[\pi_{AB}(T)]}{dT} + (\sigma_A - \sigma_B) \quad (5.7)
\]

---

4. USEFUL TECHNIQUES

1. Introduction

This chapter deals with techniques that are needed repeatedly in many experiments. While the material presented is not exhaustive it is of such a nature that it is best presented by itself, detached from specific physical experiments.

The first topic is switching circuits, since they form the basic blocks of all counting and shaping circuits used with particle detectors; moreover they are the basis of all digital logic devices such as high-speed computers. Transistors, rather than vacuum tubes, are used as switching elements. This is followed by a brief mention of electronic functional assemblies to show how the basic blocks can be put together to perform logic functions.

Next, a section on means of creating and measuring vacuum is included. For most experiments in atomic or nuclear physics a good vacuum is needed, so that it is worthwhile to discuss the capabilities and limitations of modern vacuum equipment.

Finally, a discussion of radioactive safety and handling of radioactive materials is presented. This topic has been included because in many of the following experiments, especially those on nuclear physics, the student