Nuclei and Particles

An Introduction to Nuclear and Subnuclear Physics

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Gamma rays were discovered very early among the radiations emitted by nuclei, and their electromagnetic nature was established at the same time as that of X rays (von Laue, 1912). The study of gamma rays has always played an important role in nuclear physics. They yield information on the energy and quantum numbers of nuclear states; and for this reason they are, like other nuclear radiations, a powerful tool in analyzing nuclear phenomena. In particles physics we find that some particles, the neutral pi meson, for instance, can convert themselves into pairs of gamma rays. This fact is sufficient to give considerable information about the original particle, as we shall see in Chap. 13. We must here distinguish two aspects of this study: one is essentially electromagnetic theory, the other its application to nuclear problems. In this book we are mainly concerned with the second aspect; however, we shall also treat some areas that are, in the main, the subject of electromagnetic theory. For further details and a reference book on electromagnetism consult, for example (Ja 75).

8-1 INTRODUCTION

We shall start with a semiclassical description of the radiation process. We imagine that the nucleus consists of a charge–current distribution confined to
a region about the nuclear origin and undergoing periodic motion, whose frequency $\omega$ is related to the energies involved in a nuclear transition between two levels by $\omega = (E_1 - E_2)/\hbar$. We shall, as far as is possible, apply to the radiating system concepts taken from classical electromagnetism, which we shall translate into their quantum-mechanical equivalents. A fundamental relation in electromagnetism gives the interaction between a current and the vector potential:

$$\int j_\mu A_\mu \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

where $j$ is the four vector of components $j$, $ip/c$ with $j$ and $p$ current and charge density, and $A$ is the four vector $A$, $ip$ (vector and scalar potentials) and $x_4 = ict$. This expression can be simplified and leads to the usual electric dipole formula Eq. (8-1.5), which is an approximation valid for $v \ll c$. A rigorous but less intuitive theory is based on the quantum theory of radiation.

As we shall see shortly, if the wavelength of the electromagnetic radiation considered is large compared with nuclear dimensions, the treatment is much simplified by using the "long-wavelength" approximation. This approximation is valid for energies up to several MeV, that is, for most nuclear gamma rays. However, precise quantitative calculations are seldom possible because of insufficient knowledge of nuclear wave functions. An exception is the photodisintegration of the deuteron, which will be treated in Chap. 10; nuclear photoreactions will be considered in Chap. 11. Here we shall deal primarily with emission and absorption of gamma rays occurring in transitions between nuclear levels of low or moderate excitation.

Remember that for a system of periodically moving charges, located at the origin of the coordinates, we can distinguish an induction zone defined by $r \ll \lambda = c/\omega$, in which the electric and magnetic fields can be calculated from the instantaneous velocity and position of the charges, and a "radiation" zone for $r \gg \lambda$, where retardation effects must be considered. Here $r$ is the distance from the radiator, $\lambda = 2\pi \lambda$ is the wavelength of the emitted radiation, $\omega$ is the angular frequency of motion, and $c$ is the velocity of light.

In the radiation zone, the electric and magnetic fields $\mathcal{E}$ and $\mathcal{H}$ in Gaussian units are related by the following fundamental equations:

$$|\mathcal{E}| = |\mathcal{H}| \quad \mathcal{H} \cdot \mathcal{E} = 0 \quad \mathcal{E} \cdot r = \mathcal{H} \cdot r = 0 \quad (8-1.1)$$

Moreover, they decrease as $1/r$ and give rise to a Poynting vector,

$$\mathbf{S} = \frac{\mathcal{E} \times \mathcal{H}}{4\pi} c \quad (8-1.2)$$

decreasing as $1/r^2$, which ensures a constant flow of energy toward infinity.

A detailed study of the electric dipole radiation gives the following expressions for $\mathcal{E}$ in the radiation zone: $\mathcal{E}$ is directed along the meridian, and its
magnitude is

\[ \mathcal{E}_\theta = \frac{\sin \theta}{r c^2} \vec{p} \left( t - \frac{r}{c} \right) \]  

(8-1.3)

where \( p(t - r/c) \) is the retarded value of the electric dipole moment directed in the z direction and located at the origin. The dipole varies in time according to \( p = p_0 \cos \omega t \) and \( \theta \) is the angle between \( r \) and the direction of the dipole. \( \mathcal{E} \) is equal in magnitude to \( \mathcal{E}_\theta \) and is directed along the parallels (Figs. 8-1 and 8-2). The average power radiated according to Eq. (8-1.2), with \( d\Omega \) the element of solid angle, is

\[ \langle W \rangle = \frac{c}{4\pi} \int \frac{r}{r} \cdot (\vec{E} \times \vec{H}) r^2 \, d\Omega \]

\[ = \int \frac{\sin^2 \theta}{c^2 4\pi} \langle (\vec{p})^2 \rangle \, d\Omega = \frac{2}{3} \frac{\langle (\vec{p})^2 \rangle}{c^3} = \frac{\omega^4}{3c^3} P_0^2 \]  

(8-1.4)

This is a particular case of Larmor's important formula,

\[ \langle W \rangle = \frac{2}{3} \frac{e^2 \langle a^2 \rangle}{c^3} \]  

(8-1.5)

where \( a \) is the acceleration of the charge \( e \). This formula is valid for charges moving with velocity small compared to \( c \).

Consider now two identical dipoles oriented in the z direction, having the same frequency \( \omega \) but opposite phase, and shifted with respect to each other.

**Figure 8-1** Schematic diagram of the electric dipole field in the radiation zone for a given \((r, \theta, t)\). The field has a cylindrical symmetry with respect to the z axis. The direction of the electric field is the tangent to a meridian, while the direction of the magnetic field is the tangent to a parallel for the sphere of radius \( r \).
Figure 8-2 Hertz's original figures. Electric field lines produced by an oscillating dipole at various times as a function of $r$ and $\theta$. These pictures show mainly the induction zone ($r \ll \lambda$) and the transition to the radiation zone ($r \gg \lambda$).

by a length $\Delta x \ll \lambda$. The system is called a quadrupole (Fig. 8-3). The dipole moment of the system is zero at all times, but the system nevertheless radiates energy. The field $\mathcal{E}_Q$ generated by the quadrupole at each point of space is equal to $-(\partial \mathcal{E} / \partial x) \Delta x = \mathcal{E}_Q$, where $\mathcal{E}$ is the field produced by one of the dipoles alone. It is easily seen that on the $x$ axis the quadrupole field is $\mathcal{E}(\Delta x / \lambda)$ and that the same relation obtains as an order of magnitude in all directions. The ratio of the energy radiated by our quadrupole to that radiated by one of the constituent dipoles alone is of the order of $(\Delta x / \lambda)^2$. However, the angular distribution of the intensity per unit solid angle is radically different from that of the dipole. For a nuclear system $\Delta x / \lambda$ is a number of the order of magnitude of the nuclear dimensions divided by $\lambda$. Using the relations

$$\lambda = (197 / E (\text{MeV})) \times 10^{-13} \text{ cm} \quad (8-1.6)$$
and

\[ R = 1.24^{1/3} \times 10^{-13} \text{ cm} \]

we find

\[ R/\lambda = 6.1 \times 10^{-3} 4^{1/3} E \text{ (MeV)} \]  \hspace{1cm} (8-1.7)

For low levels with \( E = 0.1 - 1 \text{ MeV} \) and for \( 4^{1/3} \) of several units, the ratio \( R/\lambda \) is of the order of \( 10^{-2} \). The quadrupole radiation is thus about \( 10^4 \) times weaker than the dipole radiation and is important only when the dipole radiation is forbidden (i.e., has intensity zero).

The important condition \( \Delta x/\lambda \ll 1 \) can be transformed to illustrate another aspect of its physical meaning. For a nucleon moving in a nucleus of radius \( R \) we have the order-of-magnitude relation

\[ \omega \approx v/R \]  \hspace{1cm} (8-1.8)

where \( v \) is the velocity of the nucleon and hence

\[ \Delta x/\lambda \approx v/c \]  \hspace{1cm} (8-1.9)

Electric dipole radiation is indicated as \( E1 \), electric quadrupole radiation as \( E2 \), and so on; in general, \( E\ell \) stands for a \( 2\ell \) pole.

In addition to the electric radiation, we must also consider magnetic multipole radiation. We may think of the magnetic dipole as a small loop of alternating current or a variable magnetic dipole. From the magnetic dipole, giving \( M1 \) radiation we pass by shifting two loops to magnetic quadrupole \( (M2) \) radiation, and so on.

In atoms the comparison between the amplitude of electric and magnetic dipole radiation gives \( E1/M1 = ea/\mu_B = "137"/2 \) where \( a \) is the Bohr radius
$\hbar^2 / me^2$ and $\mu_B$ is the Bohr magneton $\hbar/2mc$. For nuclei the amplitude of the electric dipole field produced by a charge $e$ confined to a volume $\sim R^3$ is proportional to $eR/r^3\hbar^2$ according to (8-1.3). The uncertainty relation applied to the coordinate of the charge gives

$$Rmv \approx \hbar$$  \hspace{1cm} (8-1.10)

or

$$R = \hbar / mc$$  \hspace{1cm} (8-1.11)

where $m$ and $v$ are the mass and velocity of the charge. Hence, the electric field due to the electric dipole is of the order of

$$\mathcal{E}^{(E)} \approx \frac{e\hbar}{mc} \frac{1}{r^2\hbar^2}$$  \hspace{1cm} (8-1.12)

The nuclear magnetic moment originating from the same system of charges is of the order of $e\hbar / mc$. The associated electric field is therefore proportional to

$$\mathcal{E}^{(M)} \approx \frac{e\hbar}{mc} \frac{1}{r^2 \hbar^2}$$  \hspace{1cm} (8-1.13)

$$\frac{\mathcal{E}^{(E)}}{\mathcal{E}^{(M)}} = \frac{c}{v}$$  \hspace{1cm} (8-1.14)

which means by comparison with Eq. (8-1.9) that the electric field due to the magnetic dipole is comparable to that due to the electric quadrupole. This estimate is also true for higher-order multipoles and is not altered by taking into account the intrinsic magnetic moment, which is of the same order of magnitude as the orbital one.

Returning to electric dipole radiation, we see that its emission is determined by the electric dipole moment of the radiating system, which is a vector of components

$$\Sigma ex, \Sigma ey, \text{ and } \Sigma ez$$  \hspace{1cm} (8-1.15)

where the sum is extended to all the nuclear charges. To calculate the transition probabilities quantum-mechanically, the quantities $x$ of the classical formulas must be replaced by matrix elements,

$$x \rightarrow x_{if} = \int \psi^*_f x \psi_i \, d\tau$$  \hspace{1cm} (8-1.16)

where $\psi_f$ and $\psi_i$ are the wave functions of the final and initial states, respectively.
Thus, for example, Eq. (8-1.4) gives the average power radiated by a linear dipole as

\[ \langle W \rangle = \frac{4e^2\omega^4}{3c^3} |x_{ij}|^2 \]  

(8-1.17)

Note that \( p_0^2 \) has been replaced by \( 4e^2|x_{ij}|^2 \).1 Dividing by the energy content of one quantum \( \hbar \omega \), we obtain the radiative decay constant

\[ \lambda_\gamma = \frac{4}{3\hbar} \frac{e^2}{c^3} \omega^3 |x_{ij}|^2 \]  

(8-1.18)

It is often possible to jump from one quantum-mechanical stationary state to another by different types of electromagnetic radiation, but it turns out that if \( \lambda_\gamma \gg R \), only one or two types of radiation are important. The particular type depends on the angular momentum and "parity" of the stationary states considered. We thus have selection rules, which we shall discuss shortly.

We shall now develop the classical theory by looking first at the electromagnetic field and afterward at its source.

The main purpose of the analysis of the electromagnetic fields in electric and magnetic multipoles is to separate the different parts with definite parity and angular momentum so as to be able to establish selection rules with the help of the conservation theorems. Using this approach, we derive from Maxwell's equations in vacuum the vector equations for the electric and magnetic fields. Let \( \mathcal{E} \) and \( \mathcal{H} \) be the electric and magnetic time-dependent fields,

\[ \mathcal{E}(x, y, z, t) = E(x, y, z) e^{-i\omega t} \]  

(8-1.19)

and

\[ \mathcal{H}(x, y, z, t) = H(x, y, z) e^{-i\omega t} \]  

(8-1.20)

1This substitution may be justified by the quantum theory of radiation. A heuristic argument considers large quantum numbers for which classical and quantum theory must give the same results. Classically the transition probability is

\[ \lambda_\gamma = \frac{\langle W \rangle}{\hbar \omega} = \frac{\omega^3}{3\hbar c^3} p_0^2 = \frac{2}{3} \frac{e^2 \omega E}{\hbar c^3 m} \]

In the last passage we have used the relation

\[ E = m\omega^2 p_0^2/2e^2 \]

between energy and amplitude of the dipole for an harmonic oscillator.

According to quantum theory

\[ |X_{n,n-1}|^2 = \hbar n/2m\omega \approx E/2m\omega^2 \]

and we obtain the correct result for \( \lambda_\gamma \) by replacing \( p_0^2 \) with \( 4e^2|x_{ij}|^2 \) in the classical expression for \( \lambda_\gamma \).
Maxwell's equations as shown in electricity give the relations

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \quad \text{and} \quad \nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0 \quad (8-1.21)$$

where $k = \omega / c$, with the subsidiary condition $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0$. We try to solve these equations in polar coordinates. The procedure is to obtain a complete set of solutions of the vector equations by solving first the simpler scalar equation

$$\nabla^2 \Phi_i^m + k^2 \Phi_i^m = 0 \quad (8-1.22)$$

This is accomplished by setting

$$\Phi_i^m (r, \theta, \varphi) = f_i (kr) Y_i^m (\theta, \varphi) \quad (8-1.23)$$

where $f_i (kr)$ is a Hankel function defined in terms of the ordinary Bessel function of half-odd-integer order as

$$f_i (kr) = \left( \frac{\pi}{2kr} \right)^{1/2} \left[ J_{i+1/2} (kr) + i N_{i+1/2} (kr) \right] \quad (8-1.24)$$

and $Y_i^m (\theta, \varphi)$ is the spherical harmonic. Note that the spherical harmonics form an orthonormal set obeying the relation

$$\int Y_i^{* m'} (\theta, \varphi) Y_i^m (\theta, \varphi) \, d\Omega = \delta_{i,i} \delta_{m,m'} \quad (8-1.25)$$

Application of the operators

$$\mathbf{L} = -i (\mathbf{r} \times \nabla) \quad \text{and} \quad \frac{-i}{k} (\nabla \times \mathbf{L}) \quad (8-1.26)$$

to $\Phi_i^m$ gives the vectors

$$\mathbf{F}_i^{(0)} = \mathbf{L} \Phi_i^m \quad \text{and} \quad \mathbf{F}_i^{(1)} = \frac{-i}{k} (\nabla \times \mathbf{L}) \Phi_i^m \quad (8-1.27)$$

which are solutions of Eq. (8-1.21) if we put

$$\mathbf{H}_i = -\mathbf{F}_i^{(0)} \quad \mathbf{E}_i = \mathbf{F}_i^{(1)} \quad (8-1.28)$$

for an electric multipole $Ei$ or

$$\mathbf{H}_i = \mathbf{F}_i^{(1)} \quad \mathbf{E}_i = \mathbf{F}_i^{(0)} \quad (8-1.29)$$

for a magnetic multipole $Mi$. 

The calculations necessary to show these results are found in (BW 52) and (Ja 75). Note that the operator $\mathbf{L}$ is essentially the angular momentum operator of quantum mechanics.

The relation between the fields of an electric multipole and those of a magnetic multipole is simple. We pass from one to the other by interchanging electric and magnetic fields and changing the sign of the electric field. This transformation is called a "dual" transformation and is expressed by

$$
\mathbf{E}' = -\mathbf{H} \quad \mathbf{H}' = \mathbf{E}
$$

(8-1.30)

where the primed field is the dual of the unprimed one.

We can now determine what kind of source located at the origin of the coordinates would give a field described by Eqs. (8-1.28) and (8-1.29). We would find in the case of Eq. (8-1.28) that an electric $2^l$ pole is necessary, whereas the field of Eq. (8-1.29) would be generated by a $2^l$ magnetic pole.

$\mathbf{E}l$ and $\mathbf{M}l$ fields have different symmetry properties, and this fact is important in establishing selection rules. For example, let us consider an $\mathbf{E}1$ and an $\mathbf{M}1$ field. These are generated by charges or currents, and we shall consider the fields produced at a time $t$ and at a point $\mathbf{r}(x, y, z)$ by a distribution of moving charges having the coordinates $\mathbf{s}_i$.

The fields are a function of $\mathbf{r}$ and $\mathbf{s}$, where the $\mathbf{s}$ are all the coordinates of the charges. We can thus write $\mathbf{E}(\mathbf{r}, \mathbf{s})$, $\mathbf{H}(\mathbf{r}, \mathbf{s})$. Now suppose that we change the coordinates of the charges from $\mathbf{s}$ to $-\mathbf{s}$, which means that we reflect the position of the charges with respect to the origin. For an electric dipole field we have (cf. Fig. 8-4a and b)

$$
\mathbf{E}(\mathbf{r}, \mathbf{s}) = -\mathbf{E}(\mathbf{r}, -\mathbf{s})
$$

$$
\mathbf{H}(\mathbf{r}, \mathbf{s}) = -\mathbf{H}(\mathbf{r}, -\mathbf{s})
$$

(8-1.31)

If, on the other hand, we look at the same electric dipole field at points $\mathbf{r}, -\mathbf{r}$ without changing the source, we have (Fig. 8-4a and b)

$$
\mathbf{E}(\mathbf{r}, \mathbf{s}) = +\mathbf{E}(-\mathbf{r}, \mathbf{s})
$$

$$
\mathbf{H}(\mathbf{r}, \mathbf{s}) = -\mathbf{H}(-\mathbf{r}, \mathbf{s})
$$

(8-1.32)

Fields obeying Eqs. (8-1.31) and (8-1.32) are called *odd*.

On the other hand, for a magnetic dipole field we have (cf. Fig. 8-4c and d)

$$
\mathbf{E}(\mathbf{r}, \mathbf{s}) = \mathbf{E}(\mathbf{r}, -\mathbf{s})
$$

$$
\mathbf{H}(\mathbf{r}, \mathbf{s}) = \mathbf{H}(\mathbf{r}, -\mathbf{s})
$$

(8-1.33)

because reflection through the origin does not change the sense of rotation of
Figure 8-4 Dual fields of the electric and magnetic dipoles. (a) and (b) show the electric dipole field and illustrate its symmetry properties on reflection of the source, or of the observation point through the origin. (c) and (d) show the magnetic dipole field in like manner. Comparison of (a) with (c) shows how the dual transformation changes the \( E1 \) field into the \( M1 \) field and vice versa.

The charges. Changing \( r \) to \( -r \) (Fig. 8-4c and d) gives

\[
\mathcal{E}(r, s) = -\mathcal{E}(-r, s) \\
\mathcal{K}(r, s) = +\mathcal{K}(-r, s)
\]

(8-1.34)

Fields, such as the magnetic dipole field, that obey Eqs. (8-1.33) and (8-1.34) are called even fields. From Eqs. (8-1.31) to (8-1.34) it follows that for all radiation fields, even or odd,

\[
\mathcal{E}(r, s) = -\mathcal{E}(-r, -s) \\
\mathcal{K}(r, s) = \mathcal{K}(-r, -s)
\]

(8-1.35)

Equation (8-1.35) is a consequence of the fact that the combined operation \( r \to -r, \ s \to -s \) is an inversion of all coordinates. \( \mathcal{E} \), a polar vector, then changes sign, whereas \( \mathcal{K} \), an axial vector, or pseudovector, does not. (See also Chap. 9).

From the definition of Eqs. (8-1.28) to (8-1.30) it follows that dual fields have opposite parity.

The classical electric and magnetic multipoles, located at the source, which generate the fields of Eqs. (8-1.28) and (8-1.29) have expressions of the form

\[
Q_{lm} = \int \rho(r)r^lY^m_l(\theta, \phi) \, dr \quad \text{for} \quad E1
\]

(8-1.36)
and

\[ M_{lm} = - \frac{1}{c(l + 1)} \int r' Y_l^m (\theta, \varphi) \nabla \cdot (r \times j) \, d\tau \quad \text{for} \quad Ml \quad (8-1.37) \]

where the integral number \( m, (-l \leq m \leq l) \), characterizes the \( 2l + 1 \) \( E1 \) or \( Ml \) independent multipole fields of order \( l \), \( \rho(r)e^{-i\omega t} \) is the electric charge density, \( j \) is the electric-current density at \( r \) related to \( \rho \) by

\[ i\omega \rho = \nabla \cdot j(r) \quad (8-1.38) \]

and the integral is extended to the whole region containing the charges. The continuity equation (8-1.38) connects a time variation of \( \rho \) with a current density. Hence, electric and magnetic radiation are present together. However, they cannot be of the same order, as we shall see presently.

Let us first consider electric multipole transitions. We shall assume that we have only one charged particle of coordinates \( r \). The results obtainable in this very special case are true in general. To obtain the transition probabilities in quantum mechanics, we replace \( \rho(r) \) by

\[ e \psi_i^* (r) \psi_j (r) \quad (8-1.39) \]

for the single-particle case. If \( A \) particles are involved, a procedure similar to that discussed in Sec. 6-8 is used.

In the special case of \( E1 \), it is easily recognized that Eq. (8-1.36) gives, apart from constants, the matrix elements of Eqs. (8-1.12) and (8-1.13). To see this it suffices to replace the spherical harmonics by their expressions in cartesian coordinates. This equivalence is true for any \( l \).

### 8.2 SELECTION RULES

If, for a certain order of multipole, the matrix elements that determine the transition probability vanish exactly, the transition is forbidden and that multipole component of the electromagnetic field is absent. We thus have selection rules.

A simple selection rule derives immediately from the parity properties of the function \( \psi \). For instance, for any central system we must have

\[ \psi (r) = \pm \psi (-r) \quad (8-2.1) \]

This relation results from the fact that, for a system in which the potential depends on \( |r| \) only (central system), the hamiltonian does not change if we change \( r \) into \( -r \). If a solution of the corresponding Schrödinger equation is
not degenerate, we then have

\[ H(r)\psi(r) = E\psi(r) \]
\[ H(-r)\psi(r) = E\psi(r) \]
\[ H(r)\psi(-r) = E\psi(-r) \]

which implies that \( \psi(r) = k\psi(-r) \), with \( k \) constant. If we now perform the operation of inversion (change from \( r \) to \(-r\)) twice, which obviously gives the identity, we have

\[ \psi(r) \rightarrow k\psi(-r) \rightarrow k^2\psi(r) = \psi(r) \quad \text{or} \quad k = \pm 1 \]

(8-2.3)

If \( k = 1 \), the eigenfunction is called \textit{even}, or of parity +1; if \( k = -1 \), it is called \textit{odd}, or of parity −1. If the eigenfunction is degenerate, one can always form linear combinations \( \psi(r) \pm \psi(-r) \) that have the property indicated in Eq. (8-2.3).

The product of two even functions or of two odd functions is even; the product of an even function multiplied by an odd one is odd. The integral over all space of an odd function is zero, because the contributions to the integral of elements of volume at \( r \) and \(-r\) cancel each other out.

The function \( r^lY^m_l(\theta, \phi) \) is itself even or odd, according to whether \( l \) is even or odd, irrespective of \( m \) (see Sec. 6-4).

Assume now that \( \psi_i(r) \) and \( \psi_f(r) \) have the same parity. Then the integrand in Eq. (8-1.36) will have the parity \((-1)^l\), and the integral will vanish for \( l \) odd. Similarly, if \( \psi_i \) and \( \psi_f \) have opposite parity, the integral will vanish for \( l \) even.

We thus have the selection rule that \( El \) radiation for \( l \) odd (even) is accompanied by a (no) change of parity in the eigenfunction. We shall designate by \( \Delta \pi = 1 \) (or yes), \( \Delta \pi = 0 \) (or no) a transition with or without a change of parity.

In addition to the selection rules connected with the change of parity, there are very important selection rules associated with the change of angular momentum of the nucleus. We cannot give here a general proof for these rules, but we shall give examples to show how they arise, and then state the general case.

If the nucleon intrinsic spin is neglected, the initial and final states of the nucleus have angular momenta \( l' \) and \( l'' \), respectively, with \( l \) integer, and \( z \) components of the angular momenta \( m' \) and \( m'' \), respectively. The initial and final states will thus have eigenfunctions of the form

\[ \psi_i = f_{l'm'}(r)Y^m_l(\theta, \phi) \]

(8-2.4)
and

\[ u_f = \int_{r_{, I^-}} (r) Y_{I^-}^{m^-} (\theta, \varphi) \]

(8-2.5)

If we consider electric dipole radiation (\(E1\)), the matrix elements will be given by Eq. (8-1.36) and will contain integrals of the type

\[ \int Y_{I^-}^{m^-} (\theta, \varphi) Y_{I^{'}}^{1, 0, -1} (\theta, \varphi) Y_{I^{''}}^{m^{''}} (\theta, \varphi) \, d\Omega \]

(8-2.6)

It suffices then to remember that \(Y_{I}^{0, \pm 1}\) are proportional to \(\cos \theta\), \(\sin \theta e^{\pm i\varphi}\), and the general form of \(Y_{I}^{m}\), to see immediately that the integrand will contain a factor \(e^{i(m'-m^{'})\varphi}\) or \(e^{i[(m'-m^{'})\pm 1] \varphi}\) and thus that the integral will vanish unless

\[ m' = m'' \]

(8-2.7)

or

\[ m' = m'' \pm 1 \]

(8-2.8)

Similarly one finds that the integral vanishes also unless

\[ l' = l'' \pm 1 \]

(8-2.9)

by using properties of \(Y_{I}^{m}(\theta, \varphi)\) mentioned in Problem 8-8. Now, if we recall the meaning of \(l\) and \(m\), we see that in the emission of electric dipole radiation the total angular momentum of the nucleus changes by one unit and that its \(z\) component changes by zero or one unit. This argument may be extended to noninteger spin and arbitrary radiation, with the result that for \(E1\) radiation we must have

\[ |J' - J''| \leq l \leq J' + J'' \]

(8-2.10)

In this chapter \(J', J''\) are the total angular momentum in the initial and final nuclear states. [For the proof refer to (Ma 57) or (BW 52).] Equation (8-2.10) has the geometrical interpretation that it must be possible to construct a triangle of sides \(J', J'',\) and \(l\). The principle of conservation of angular momentum suggests that the outgoing wave of \(E1\) light carries an angular momentum of magnitude \(l\) with respect to the center of mass of the nucleus. This is borne out by direct calculation of the density of angular momentum for the quantized electromagnetic field.

Thus far we have been concerned only with electric multipoles. Magnetic multipoles arise from the electric currents caused by the motions of the charges in the nucleus and from the intrinsic magnetic moments connected
with the spins. Remember in this connection that the neutron, although neutral, has a magnetic moment. In the simplest case of the dipole due to the motion of a spinless charged particle, the z component of the matrix element for magnetic radiation is proportional to

$$\int \psi_i^* (x \rho_\perp - y \rho_\perp) \psi_i \, d\tau$$  \hspace{1cm} (8-2.11)

because the operator $x \rho_\perp - y \rho_\perp = L_z$ is proportional to the component of the magnetic moment produced by the electric current associated with the transition. This is to be compared to the electric dipole matrix element given by Eq. (8-1.16). We shall shortly see the effect of the intrinsic magnetic moment.

In the specific case mentioned above, the $M1$ matrix element is not zero only if $\psi_i^*, \psi_i$ have the same parity, because the operator $L_z$ does not change the parity of $\psi_i$. Moreover, as with $E1$ transitions, selection rules for $M1$ are $\Delta l = \pm 1$ and $\Delta m = \pm 1, 0$, as can be easily verified.

The selection rules for magnetic radiation can be generalized and can be interpreted as representing conservation of angular momentum if one associates with $M1$, $M2$, $M3$ radiation 1, 2, 3 units of angular momentum, just as in the case of $E1$, $E2$, and $E3$. In fact, from the relation between the fields of $E1$ and $M1$ expressed by Eqs. (8-1.28) and (8-1.29) and the expression for the density of momentum $p$ in an electromagnetic field,

$$p = \frac{\mathcal{E} \times \mathcal{H}}{4\pi c}$$  \hspace{1cm} (8-2.12)

we see that the magnitude of the density of momentum of $E1$ and $M1$ is the same, and hence the magnitude of the density of angular momentum is also identical in the two cases.

In Table 8-1 we sum up the results for the lowest-order radiation possible between two states of angular momentum $J', J''$ and parity $\pi'$, $\pi''$. It must be remembered that the triangular relation [Eq. (8-2.10)] must always be obeyed. Hence, the table gives conditions that are necessary but not sufficient for radiative transitions. Thus, for example, transitions from $J' = 0$ to $J'' = 0$ are always forbidden, and transitions $J' = \frac{1}{2}$ to $J'' = \frac{1}{2}$, $\Delta \pi = 0$ cannot occur as

<table>
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<tr>
<th>TABLE 8-1 SELECTION RULES FOR ELECTROMAGNETIC MULTIPOLE RADIATION</th>
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<td>$E1$</td>
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8.3 TRANSITION PROBABILITIES

$E2$, because the triangular relation is violated. Transitions from $J' = 0$ to $J'' = 0$ can occur only by mechanisms different from electromagnetic radiation, namely, by the emission of conversion electrons or by the formation of electron–positron pairs.

In practice the types of radiation that have been observed up to the present are $E1$ to $E6$ inclusive and $M1$ to $M5$ inclusive. In almost all cases, except the pairs $E2–M1$ and $E1–M2$, only a single type of radiation occurs in a given transition.

8.3 TRANSITION PROBABILITIES

If we generalize the results obtained in Sec. 8.2, the structure of the complete formulas for the transition probabilities will appear plausible. The detailed calculation (BW 52) yields

$$\lambda^{(E)}(l, m) = \frac{8\pi(l + 1)}{\hbar l[(2l + 1)!!]^2} \left( \frac{\omega}{c} \right)^{2l+1} \left| Q_{lm} + Q_{lm}' \right|^2 \tag{8-3.1}$$

$$\lambda^{(M)}(l, m) = \frac{8\pi(l + 1)}{\hbar l[(2l + 1)!!]^2} \left( \frac{\omega}{c} \right)^{2l+1} \left| R_{lm} + R_{lm}' \right|^2 \tag{8-3.2}$$

Here $l$ is the order of the transition, and $n!! = 1 \cdot 3 \cdot 5 \cdots n$. The first formula is valid for $E1$ and the second for $M1$ radiation. $Q_{lm}$ and $R_{lm}$ are the parts of the matrix element containing the ordinary coordinates, and $Q_{lm}'$ and $R_{lm}'$ are the parts of the matrix elements containing the intrinsic magnetic moment.

We can write them formally as

$$Q_{lm} = e \sum_{k}^{Z} \int r_k' Y_{l}^{m*}(\theta, \varphi_k) \psi_{l}^{*} \psi_i \: d\tau \tag{8-3.3}$$

$$R_{lm} = -\frac{1}{l + 1} \frac{e\hbar}{2Mc} \sum_{k}^{Z} \int r_k' Y_{l}^{m*}(\theta, \varphi_k) \nabla \cdot (\psi_{l}^{*} L_{k} \psi_i) \: d\tau \tag{8-3.4}$$

$$Q_{lm}' = -\frac{i(\omega/c)}{l + 1} \frac{e\hbar}{2Mc} \sum_{k}^{A} \int \mu_k r_k' Y_{l}^{m*}(\theta, \varphi_k) \nabla \cdot (\psi_{l}^{*} r_k \times \sigma_k \psi_i) \: d\tau \tag{8-3.5}$$

$$R_{lm}' = -\frac{e\hbar}{2Mc} \sum_{k}^{A} \int \mu_k r_k' Y_{l}^{m*}(\theta, \varphi_k) \nabla \cdot (\psi_{l}^{*} \sigma_k \psi_i) \: d\tau \tag{8-3.6}$$

Here the symbols have the following meanings: $i$ and $f$ indicate initial and final states; $k$ is the number of the nucleon; 1 to $Z$ for protons, $Z + 1$ to $A$ for
neutrons; \( \psi \) refers to the eigenfunction of the whole nucleus; the vector operator \( \mathbf{L}_k \) is \(-i\mathbf{r}_k \times \nabla_k\); the \( \mu_k \) are the magnetic moments of the nucleons in units of \( e\hbar/2Mc \); the \( \sigma \) are Pauli matrix operators. The spherical harmonics are evaluated for the position of each nucleon in turn. The \( \lambda(l, m) \) averaged over the initial \( m' \) states and summed over the final \( m'' \) states correspond to the average transition probability from an unpolarized source irrespective of the polarization of the emitted radiation. We call

\[
B(l, J_i, J_f)
\]

the expression

\[
\frac{1}{2J_i + 1} \sum_{m'} \sum_{m''} | \psi_{lm} + \psi_{lm}'|^2
\]

or

\[
\frac{1}{2J_i + 1} \sum_{m'} \sum_{m''} | \chi_{lm} + \chi_{lm}'|^2
\]

averaged and summed up as indicated. We introduce correspondingly the expression

\[
\lambda(l) = \frac{1}{2J_i + 1} \sum_{m'} \sum_{m''} \lambda(l, m)
\]

and we have

\[
\lambda(l) = \frac{8\pi(l + 1)}{l[(2l + 1)!!]^2} \left( \frac{\omega}{c} \right)^{2l + 1} \frac{1}{\hbar} B(l, J_i \rightarrow J_f)
\]

If we need to distinguish electric and magnetic transitions, we shall use \( \lambda(El) \), \( B(El) \), etc.

The evaluation of formulas such as Eqs. (8-3.3) to (8-3.11) requires a detailed knowledge of the nucleus. Detailed calculations are possible only for low-lying states and for some simple models; a notable case is that of only one nucleon radiating. In particular we shall consider a nucleus of odd \( A \) according to the shell model. Its angular momentum is then due to the odd nucleon alone, and we assume \( j-j \) coupling. The radiation is emitted only because the single nucleon changes orbit. We write the interesting part of the eigenfunction as

\[
\psi_i = R_{nl'}(r)\phi_{j', l', m'}(\theta, \varphi)
\]

and a similar one for \( \psi_f \).
By inserting Eq. (8-3.12) into the multipole formulas (8-3.3) to (8-3.6), the parts containing the spherical harmonics can be integrated and factored out. They give a result that we indicate by

\[ S(J_i, J_f, l) \]

(8-3.13)

The numerical value of this expression has been explicitly calculated (Moszkowski, 1951) and tabulated. In general, it is of the order of magnitude of unity. The matrix element \( \mathcal{O}'_{lm} \), \( \mathcal{R}'_{lm} \), and \( \mathcal{R}''_{lm} \) have the dimensions of \( er^l \) and \( (eh/Mc)r^{l-1} \), respectively. The integrals expressing them are extended over the nuclear volume; thus the variable \( r \) appearing in them has the order of magnitude of the nuclear radius \( R \). This suggests introducing dimensionless quantities

\[ \mathcal{O}(E_l, J_i, J_f) = (\mathcal{O}_{lm} + \mathcal{O}'_{lm})/eR^l \]

and

\[ \mathfrak{M}(M_l, J_i, J_f) = (\mathcal{R}_{lm} + \mathcal{R}'_{lm})/[(eh/Mc)R^{l-1}] \]

summed over \( m'' \) and averaged over \( m' \). One then obtains for the transition probabilities

\[ \lambda(E_l) = \left( \frac{e^2}{\hbar c} \right) \frac{l+1}{l} \frac{\omega}{[(2l+1)!!]^2} \left( \frac{\omega R}{c} \right)^{2l} (2J_f + 1) S|\mathcal{O}(E_l)|^2 \]

(8-3.14)

and

\[ \lambda(M_l) = \left( \frac{e^2}{\hbar c} \right) \frac{l+1}{l} \frac{\omega}{[(2l+1)!!]^2} \left( \frac{\omega R}{c} \right)^{2l} (2J_f + 1) S \left( \frac{\hbar}{McR} \right)^2 |\mathfrak{M}(M_l)|^2 \]

(8-3.15)

In the case in which a single proton changes state in the transition,

\[ \mathcal{O}(E_l) = \int_0^\infty R_i(r) \left( \frac{r}{R} \right)^l R_f^*(r) r^2 dr \]

(8-3.16)

and

\[ \mathfrak{M}(M_l) = \left( \mu_p l - \frac{l}{l+1} \right) \int_0^\infty R_i(r) \left( \frac{r}{R} \right)^{l-1} R_f^*(r) r^2 dr \]

(8-3.17)

where \( \mu_p = 2.79 \), the magnetic moment of the proton in nuclear magnetons.
Figure 8.5 The transition probability for gamma-ray emission (as a function of the transition energy $E_{\gamma}$ in MeV) based on the single-particle model. Part (a) plots the transition probability for $E1$ radiation ($l = 1, \ldots, 5$) for nuclei of mass 20, 50, 130, and 220 according to the formula

$$\lambda(E1) = \frac{4(l + 1)}{l[(2l + 1)!!]} \left( \frac{3}{3 + l} \right)^2 \left( \frac{E_{\gamma}}{140} \right)^{2l+1} A^{2l/3} \frac{mc^2}{\hbar}$$
Part (b) plots the transition probability for $M^l$ radiation ($l = 1, \ldots, 5$) according to the formula

$$\lambda(M^l) = \frac{0.088(l + 1)}{l!(2l + 1)!} \left( \frac{3}{2 + l} \right)^2 \left( \frac{E_\gamma}{140} \right)^{2l+1} A^{(2l-2)/3} \left( \frac{\mu_p l}{2} - \frac{l}{l + 1} \right)^2 \frac{mc^2}{\hbar}$$

The formulas neglect the factor $S(J_f, J_i, l)$ of Eq. (8-3.13) and assume

$$R = 1.40A^{1/3} F$$

The curves represent the additional contributions to the total transition probability by the internal conversion process. [From E. U. Condon and H. Odishaw, *Handbook of Physics*, McGraw-Hill, New York 1967.]
and $R_i$ and $R_f$ are the radial eigenfunctions of the initial and final states. For the single-neutron case

$$\Sigma(El) = 0$$  \hspace{1cm} (8-3.18)

because the neutron has no charge, and

$$\mathcal{M}(Ml) = \mu_n l \int R_i(r) \left(\frac{r}{R}\right)^{l-1} R^*_f(r) r^2 \, dr$$  \hspace{1cm} (8-3.19)

The radial integrals may be approximated by assuming $R_n(r) = \text{constant} = (3/R^3)^{1/2}$ from $r = 0$ to $r = R$, as required by normalization, and zero for $r \gg R$. We then have immediately, for example,

$$\Sigma(El) = \int R^*_f(r) \frac{r^l}{R^l} R_i(r) r^2 \, dr = \frac{3}{3 + l}$$  \hspace{1cm} (8-3.20)

and similar expressions for the other matrix elements.

The transition probabilities have been evaluated on the usual assumption that $R = r_0 A^{1/3}$. They are shown in Fig. 8-5. The transition probabilities depend strongly on the energy of the transition through the factor $\omega^{2l+1}$ and on $A$, which enters through the radius $R$ at the power $2l$. The approximations involved are crude, and one cannot expect good numerical agreement. Important effects such as the recoil of the rest of the nucleus except the nucleon considered have been entirely neglected, and an oversimplified model has been used. We gain an impression of the measure of agreement between the experimental results and our schematization, comparing the values of the experimental mean life for gamma transitions with the values predicted by Eqs. (8-3.14) and (8-3.15). To facilitate the comparison, we use the "reduced mean life," or "comparative mean life," that is, the mean life corrected by the factors due to transition energy and nuclear size (Fig. 8-6).

Experimentally the mean life of a gamma emission can be measured directly down to approximately $10^{-10}$ sec. Indirect measurements involve the observation of Coulomb excitation (see Sec. 8-7) and the observation of level width either through resonance fluorescence or otherwise. Here one can reach values of about $10^{-12}$ sec. The direct observation of the level width is a valid method for extremely short times. In particle physics, levels of several MeV in width are common and 1 MeV corresponds to $6.58 \times 10^{-22}$ sec.

To show the influence of the type of motion on the gamma transition probabilities, we shall mention a case almost opposite to the one-particle model: the liquid-drop model. According to this model, neutrons and protons in a nucleus are bound in such a way that the local composition of nuclear matter is practically constant. The electric center of charge then has the same coordinates as the center of mass and cannot move under the action of internal forces, because of the principle of the conservation of momentum. It
Figure 8.6 Comparison of experiment with the "reduced half-life" of gamma transition. The "reduction" should eliminate the influence of energy and ΔI. [M. Goldhaber and J. Weneser. *Ann. Rev. Nucl. Sci.*, 5, 1 (1955).] The various transitions are expected to fall in the regions labeled E2, E3, etc. at the right.
follows that the electric dipole moment is zero, and dipole radiation is strictly forbidden. This situation exists in many low-energy transitions in which the dipole radiation is much weaker than would be expected on the basis of Eq. (8-3.14).

At high energy (∼ 20 MeV) there seems to be a different type of motion, in which all protons together oscillate relative to all neutrons. This motion gives rise to a large electric dipole moment that enables the nuclei to absorb electromagnetic energy strongly in that frequency range and hence gives rise to large cross sections for the (γ, n), (γ, p) reactions. The absorption has the character of a broad resonance, sometimes called the giant resonance, and will be treated in Chap. 11.

Finally, special types of surface motion in the nucleus occasionally favor electric quadrupole radiation (see Secs. 6-12 and 8-7).

### 8-4 INTERNAL CONVERSION

In the preceding section we have considered the transition of a nucleus from one level to another by emission of electromagnetic radiation. This and, given sufficient energy, pair production would be the only ways to execute the transition for an isolated nucleus deprived of all its atomic electrons. The presence of the electrons makes possible a different process: the nucleus can lose its excitation and transfer it directly to one of the atomic electrons which is ejected with a kinetic energy equal to the energy of the gamma transition minus the binding energy of the electron. Electrons ejected by this mechanism are called conversion electrons, and one speaks of conversion in the K shell, in the L shell, etc., according to the shell vacated by the conversion electron. The conversion coefficient

\[
\frac{N_e}{N_\gamma} = \alpha \tag{8-4.1}
\]

is the ratio between the average number of electrons and the average number of gamma rays emitted in connection with a given transition. It is possible also to distinguish partial conversion coefficients according to the shell from which the electron is taken. Thus one has

\[
\alpha_K + \alpha_L \cdots = \alpha \tag{8-4.2}
\]

where \(\alpha_K, \alpha_L\), etc., are the partial conversion coefficients (Fig. 8-7).

The possibility of decay of an excited state by internal conversion adds to its decay constant; thus the total decay constant \(\lambda\) is equal to the sum of the partial decay constants \(\lambda_\gamma\) and \(\lambda_e\) for gamma emission and conversion electron emission. As a consequence of Eq. (8-4.1) one has

\[
\lambda = \lambda_\gamma + \lambda_e = \lambda_\gamma (1 + \alpha) \tag{8-4.3}
\]
Figure 8-7 Internal conversion spectrum of $^{209}$At, $^{210}$At, $^{211}$At, showing $K$, $L$, $M$, and $N$ conversion. Numbers are energies in KeV. [J. W. Mihelich, A. W. Schardt, and E. Segrè, *Phys. Rev.*, 95, 1508 (1954).]
A direct experimental proof of the last relation is given by the decay of an excited state in $^{99}$Tc, which proceeds with a slightly longer mean life if some of the surrounding electrons are removed by chemical bonding.

If there are two radioactive emissions in rapid succession, e.g., $\beta$ (negative electrons) and gamma emission, the energy of the conversion electrons tells which emission is the first. Starting with a nucleus of atomic number $Z$, if the beta emission precedes the gamma emission, the binding energies of the conversion electrons correspond to an atom with atomic number $Z + 1$ and not $Z$ (L. Meitner and H. J. von Baeyer, 1919). Similar considerations obtain for alpha emission or orbital electron capture associated with gamma emission.

The internal conversion coefficients depend on the atomic number of the nucleus, on the energy, and on the character of multipolarity of the transition, but not on the specific nuclear model. Thus their study is a powerful aid to the classification of nuclear levels. Their values as a function of the energy, of the type of radiation, and of $A$ have been extensively tabulated. We shall now calculate one of the simplest possible cases, which will bring out the essentials of the phenomenon.

Suppose that the nucleus is in an excited state from which it can pass to the ground state by the emission of $E1$ radiation. The nucleus can then be compared with an electric dipole of frequency $\omega$. The presence of this dipole may induce transitions from the ground state of the atom to an excited state; specifically, the $K$ electrons, which are in a $1s$ state, can be brought by dipole radiation to a $p$ state, possibly in the continuum. The probability of this transition is calculable with golden rule No. 2,

$$w = \frac{2\pi}{\hbar} |M_{ij}|^2 \frac{d\mathcal{R}}{dE} \tag{8-4.4}$$

We have to evaluate the matrix element $M_{ij}$ and the density of the accessible final states. Call the initial eigenfunction of the electron in the $1s$ state

$$\psi_i(r, t) = u_i(r) \exp\left(-\frac{iE_i}{\hbar} t\right) \tag{8-4.5}$$

and the final eigenfunction of the electron in the continuum

$$\psi_f(r, t) = u_f(r) \exp\left(-\frac{iE_f}{\hbar} t\right) \tag{8-4.6}$$

The transition from the initial to the final state is induced by the electric field of the nucleus, which is described as an electric dipole of moment $\mathbf{P}$ directed along the $z$ axis and varying in time with frequency $\omega$. The electric potential of this dipole is

$$V(r, t) = p_0 \frac{\cos \theta}{r^2} \cos \omega t = p_0 \frac{\cos \theta}{r^2} \frac{1}{2} (e^{i\omega t} + e^{-i\omega t}) \tag{8-4.7}$$
where $\theta$ is the angle between $\mathbf{r}$ and the $z$ axis. The matrix element of the induced transitions is

$$M_{ij} = e \int \psi_i^* (\mathbf{r}, t) V \psi_j (\mathbf{r}, t) \, d\tau$$  \hspace{1cm} (8-4.8)

$M_{ij}$ is of appreciable magnitude only if

$$|E_i - E_j| = \hbar \omega$$  \hspace{1cm} (8-4.9)

Moreover, since $u_i$ corresponding to an $s$ state does not contain $\theta$, while $V$ contains the factor $\cos \theta = P_1 (\cos \theta)$, $\psi_i^*$, when expanded in spherical harmonics, will contribute to the integral only through the term that also contains $P_1 (\cos \theta)$, all the other terms being orthogonal to $u_i V$. This shows that the transition will occur only to $p$ levels. The density of the final states in Eq. (8-4.4) must thus be limited to the density of the $p$ states.

At this point we can already see the important qualitative conclusion mentioned above. The probability of internal conversion and the radiation probability are both proportional to $p_0^2$, hence this quantity will disappear from the internal conversion coefficient. The internal conversion coefficient is a function of the energy of the transition and of the atomic number of the atom, because they appear in the initial and final wave functions, but not of the complicated nuclear wave functions. For other multipole fields the same qualitative conclusion obtains: the internal conversion coefficients depend on the character of the radiation ($E1$ or $M1$), the atomic shell in which it occurs, the atomic number, and the energy.

Note that this result is not absolutely exact. There are some additional effects depending on nuclear size and internal motions, which in special cases affect the conversion coefficient appreciably. In the main, however, it is correct to assume that the internal conversion coefficient is a purely atomic property.

In order to take the calculation a little further in our simple example, we take as the wave function of the final state that of a free electron and expand it in spherical harmonics (see Appendix A). We are, however, interested only in the $p$-wave component, because all other components give zero matrix elements with the initial $s$ state. We thus write

$$u_f = N \frac{\cos \theta}{(kr)^{1/2}} J_{3/2} (kr)$$  \hspace{1cm} (8-4.10)

or, asymptotically for large $kr$,

$$u_f = - N \cos \theta \left( \frac{2}{\pi kr^2} \right)^{1/2} \cos kr$$  \hspace{1cm} (8-4.11)
We have indicated by $N$ a normalization factor, which we evaluate by enclosing the system in a very large sphere of radius $R$ and using the asymptotic expression for the eigenfunction. We find

$$N = k \left( \frac{3}{4R} \right)^{1/2}$$  \hspace{1cm} (8-4.12)

For the initial state we take the hydrogen-type $s$-wave function

$$u_i = \frac{1}{\pi^{1/2}} \left( \frac{Z}{a_0} \right)^{3/2} \exp \left( - \frac{Zr}{a_0} \right) \text{ with } a_0 = \frac{\hbar^2}{me^2}$$  \hspace{1cm} (8-4.13)

The matrix element is then

$$M_{if} = p_0 (\cos \omega t) ek \left( \frac{3}{4R} \right)^{1/2} \frac{1}{\pi^{1/2}} \left( \frac{Z}{a_0} \right)^{3/2}$$

$$\times \int_0^\infty \exp \left( - \frac{Zr}{a_0} \right) \frac{\cos \theta}{r^2} \frac{J_{3/2}(kr)}{(kr)^{1/2}} \cos \theta \, d\tau$$  \hspace{1cm} (8-4.14)

$$= p_0 (\cos \omega t) \left( \frac{4\pi}{3R} \right)^{1/2} ek \left( \frac{Z}{a_0} \right)^{3/2} I$$  \hspace{1cm} (8-4.15)

with

$$I = \int_0^\infty \exp \left( - \frac{Zr}{a_0} \right) \frac{J_{3/2}(kr)}{(kr)^{1/2}} \, dr$$  \hspace{1cm} (8-4.16)

The density of the final states must be limited to $p$ states only. From the asymptotic expression [Eq. (8-4.11)] and the condition $u_f(R) = 0$ we find the quantization condition

$$kR = (n + \frac{1}{2})\pi$$  \hspace{1cm} (8-4.17)

with $n$ an integral number. Thus in the $k$ interval $\Delta k$ there are

$$R \frac{\Delta k}{\pi} = \Delta \mathcal{N}$$  \hspace{1cm} (8-4.18)

states. From this equation we get

$$\rho = \frac{d\mathcal{N}}{dE} = \frac{R}{\hbar \pi \nu}$$  \hspace{1cm} (8-4.19)
Taking into account the statistical weights of the $p$ states and the averaging over $M_{ij}$ required in Eq. (8-4.4), and combining Eqs. (8-4.15) and (8-4.19), we obtain for two $K$ electrons

$$\lambda_\gamma = \frac{14\pi}{\hbar} p_0^2 \frac{e^2 k^2}{3} \left( \frac{Z}{a_0} \right)^3 \frac{I^2}{\hbar \nu}$$  \hspace{1cm} (8-4.20)

On the other hand $\lambda_\gamma$ is given by

$$\lambda_\gamma = \frac{1}{3} \frac{p_0^2 \omega^3}{\hbar c^3}$$  \hspace{1cm} (8-4.21)

according to Eq. (8-1.4) and hence the internal conversion coefficient is

$$\alpha = \frac{4\pi}{\hbar} \frac{k^2 e^2}{v} \left( \frac{Z}{a_0} \right)^3 \frac{e^3}{\omega^3} I^2$$  \hspace{1cm} (8-4.22)

A closed-form evaluation can be obtained in the special case of $a_0/Z \gg 1/k$, which means that the energy of the transition is very large compared with the electron binding energy. We shall assume, moreover, that the ejected electron is not relativistic. To be consistent, then, we assume for the electron that $mv^2/2 \approx (\hbar k)^2/2m = \hbar \omega$.

The integral $I$ can be calculated by elementary means on the assumption that $e^{-Zr/a_0} = 1$, and we have

$$I = \int_0^\infty J_{3/2}(kr) \frac{dr}{(kr)^{1/2}} = \left( \frac{2}{\pi k^2} \right)^{1/2}$$  \hspace{1cm} (8-4.23)

Replacing in Eq. (8-4.22) with the approximations mentioned above, we have

$$\alpha_K = \frac{8}{\hbar} \frac{e^2 m^{1/2}}{(2\hbar \omega)^{1/2}} \left( \frac{Z}{a_0} \right)^3 \frac{c^3}{\omega^3}$$

$$= \frac{1}{2} Z^3 \left( \frac{e^2}{\hbar c} \right)^4 \left( \frac{2mc^2}{\hbar \omega} \right)^{7/2}$$  \hspace{1cm} (8-4.24)

This formula, valid under the hypothesis mentioned for dipole radiation, may be extended to $El$ radiation, giving

$$\alpha_K' = Z^3 \left( \frac{e^2}{\hbar c} \right)^4 \frac{l}{l+1} \left( \frac{2mc^2}{\hbar \omega} \right)^{l+(5/2)}$$  \hspace{1cm} (8-4.25)
The approximations used here are too crude to give valuable numerical results. However, it is possible to obtain good accuracy by employing relativistic wave functions and other necessary refinements. The extension to higher electric and magnetic multipoles becomes increasingly cumbersome. Typical numerical results are shown in Figs. 8-8, 8-9, and 8-10. Extensive tables of internal conversion coefficients are reported in the literature, for instance in (Se 59) (LHP 67), and (Si 65).

![Graph showing conversion coefficients](image)

**Figure 8-8** Electric ($\alpha_L$) and magnetic ($\beta_L$) conversion coefficients for the K shell and for $Z = 64$. Energy scale gives $E/mc^2 = k$. [From (Si 65).]

The measurement of internal conversion coefficients is performed by counting the number of gamma rays, for instance, with a scintillation counter, and the number of conversion electrons with a Geiger–Müller counter or photographically, often with the help of beta spectrographs (Fig. 8-10). The measurement of the ratios of the conversion coefficients for the different X-ray levels $\alpha_K : \alpha_{L_1} : \alpha_{L_{II}}$, etc., can be made with a beta spectrometer without
measuring gamma rays. The ratios by themselves give valuable information for classifying the type of radiation.

In some cases ($J = 0 \rightarrow J = 0$ transitions), gamma emission is forbidden in all orders and the emission of atomic electrons or electron–positron pairs is the deexcitation mechanism.

8.5 NUCLEAR ISOMERISM

The selection rules described in Sec. 8-2 can slow down electromagnetic transitions to such a point that the excited state has a very long mean life, "very long" meaning from 0.1 sec to years. In this case, the excited state is
Figure 8-10 Electric conversion coefficients for the $L_{II}$ subshell, $Z = 85$. Energy scale in $E/mc^2$. [From (Si 65).]

Figure 8-11 Types of isomerism. In (a) the upper level decays prevalently by gamma emission to the ground state, which decays by beta emission. In (b) upper and lower states decay independently by beta emission.
Figure 8-12 Energy-level diagram for $^{80}$Br, illustrating isomeric transition. All energies in MeV.
called a metastable or an isomeric state in analogy with the chemical isomers. It is clear from the definition that the limit of 0.1 sec is completely arbitrary. There are gamma transitions of mean lives ranging from $10^{-16}$ to $10^8$ sec: thus the point at which one starts to call a state metastable is arbitrary.

The phenomenon of nuclear isomerism was discovered in $^{234}$Pa by O. Hahn (1921). The explanation in terms of forbidden gamma transitions is due to von Weizsäcker (1936).

Often nuclear isomerism accompanies beta transitions, as indicated schematically in the typical level diagrams of Fig. 8-11. In Fig. 8-11a the isomeric transition between levels $a$ and $b$ is very probable compared with the beta transition between $a$ and $c$. If $\lambda_a \ll \lambda_{b\beta}$, and $\lambda_{a\beta} \gg \lambda_{\beta\alpha}$, the substance

![Figure 8-13](image-url) Distribution of long-lived isomers of odd mass number $A$ plotted against the number of odd nucleons ($N$ or $Z$). (Isomer islands.) [M. Goldhaber and J. Weneser, *Ann. Rev. Nucl. Sci.*, 5, 1 (1955).]
exhibits the beta spectrum typical of level $b$, with the decay constant $\lambda_{\beta}$. $^{80}$Br is an example, and its level diagram is illustrated in Fig. 8-12. In other cases (Fig. 8-11b) levels $a$ and $b$ decay as independent substances.

Isomeric transitions, being highly forbidden, must correspond to large $\Delta J$ and small energy. Both circumstances favor high internal conversion, and this is another characteristic of isomeric transitions. In fact, the large internal conversion may be used to separate nuclear isomers chemically. In the case of $^{80}$Br, for instance, the 4.4-h excited level emits a 49-keV gamma ray, which is highly converted and leaves in an inner shell a vacancy subsequently filled by an outer electron. This process continues until one of the valence electrons is used. If the $^{80}$Br atom is bound in an organic compound, loss of the valence electron sets it free as a Br$^-$ ion, which can be chemically separated by precipitation of AgBr. It is thus possible to separate the nuclei that have undergone isomeric transition from the others.

To be metastable an excited level must differ from lower energy levels by three or more units of $J$: the radiation emitted is thus $E3$, $M3$, or of higher multipolarity. The condition mentioned is satisfied only for $A \geq 39$, and there are no isomers of the light elements. Even at higher $A$, nuclear isomeric states are not spread uniformly among all nuclei but are preferentially concentrated in “islands” of nuclei with $Z$ or $N$ just below the magic numbers 50, 82, and 126, and even $A$. Isomers with both $N$ and $Z$ even are very rare (Fig. 8-13).

These facts are accounted for satisfactorily by the shell model. First, in an even-even nucleus, the excitation of a nucleon involves the pairing energy, which is too large to allow the formation of isomers. Second, the islands of isomerism are explained by a study of Fig. 6-36. For low $A$ up to 40 nucleons the orbits involved have $j \leq 5/2$, and there are no possibilities of large spin differences between energetically close orbits. Shortly before the numbers 50, 82, or 126, there are energetically neighboring orbits with $j = 1/2$, 9/2; 1/2, 11/2; 1/2, 13/2; and these give rise to the “islands of isomers.”

8.6 ANGULAR CORRELATIONS IN GAMMA EMISSION

Often, when two gamma rays are emitted in rapid succession by the same nucleus, one finds that the directions of emission of the two rays are correlated. This means that upon assuming the direction of the first gamma as the $z$ axis, the probability of the second falling into the element of solid angle $d\omega$ at an angle $\theta$ with the $z$ axis is not constant but depends on the angle $\theta$. This type of correlation, already mentioned in Chap. 6, is not restricted only to $\gamma-\gamma$ emission. The principles that we shall repeat here may be generalized to other cases.

Suppose that we have nuclei emitting light quanta through electric dipole transitions. We place the nuclei in a magnetic field, which orients them in such a way that the electric dipole is along the $z$ axis. We know that no quanta will be emitted in the direction of the $z$ axis and that the maximum
probability of emission will be in the $x$–$y$ plane. Conversely, in the absence of an orienting field, the fact that a gamma quantum is emitted in a certain direction tells us that the direction is not the direction of the electric dipole and makes it a priori more probable that the electric dipole is perpendicular to the direction of the first quantum. A second quantum is thus less likely to be emitted in a direction perpendicular to the first quantum than in any other. This argument can be made quantitative. The probability of finding the angle $\theta$ between successively emitted gamma rays is

$$P(\theta) \, d\omega = A (1 + \cos^2 \theta) \, d\omega$$ \hspace{1cm} (8-6.1)

where $d\omega$ is the element of solid angle and

$$A = \frac{3}{16\pi}$$ \hspace{1cm} (8-6.2)

is a normalization constant.

The quantum-mechanical treatment of the correlation for more complicated cases shows that $P(\theta)$ is a polynomial of even degree in $\cos \theta$. This is apparent from the fact that $P(\theta)$ and $P(\pi - \theta)$ must be equal if parity is conserved, that is, if the correlation is not altered on reflection through the origin, as is the case for electromagnetic interactions. The degree of the polynomial in $\cos^2 \theta$ and its coefficients depend on the spins of the three states involved and on the character of the two radiations connecting them.

![Diagram](Image)

**Figure 8-14** Apparatus for measuring angular correlations of gamma rays. PA, preamplifiers; A, amplifiers; C, coincidence circuit. 0.1-μsec resolution: S512, scale of 512; S8, scale of 8; Cr, crystals; S, source.
Angular correlations are not limited to $\gamma-\gamma$ cascades. They are observed in beta emission followed by gamma emission, in gamma emission followed by emission of Auger electrons, and in other cases. The subject has become vast, but the formulas giving the correlations are similar in all cases. For a detailed treatment see (Si 65).

In Chap. 6 we discussed the effect of a magnetic field on angular correlations. A typical apparatus is illustrated in Fig. 8-14. Despite the apparent simplicity of the equipment, considerable care must be exercised in order to avoid the numerous sources of error, such as scattering of radiation from one counter into the other, finite solid angles, and the efficiency of counters. (See also Fig. 6-32).

8.7 COULOMB EXCITATION

Nuclear levels can be excited by bombarding nuclei with charged heavy particles such as protons or alpha particles. This method of study is experimentally preferable to the use of electrons where there are background difficulties from the bremsstrahlung. Keeping the energy of the heavy charged particles below the nuclear Coulomb barrier, we avoid specific nuclear reactions, and the backgrounds due to unwanted effects are small. Projectiles used have some MeV of energy; the levels excited, some tenths of MeV of energy. The electric field of the projectile induces transitions in the target, and the effect can be calculated semiclassically by considering the field produced by the projectile moving in its hyperbolic trajectory as a perturbation on the nucleus. The matrix elements involved are the multipole electric moments of the nucleus connecting the initial and final states. The calculation is rather involved, but the underlying physical concepts are simple. For the method of calculation to be valid, the orbit concept must be applicable; hence the de Broglie wavelength $\lambda$ of the projectile must be small compared with the distance of closest approach to the target. Now calling $a$ half this distance, one has

$$a = \frac{Z_1 Z_2 e^2}{mv^2} \quad (8-7.1)$$

where $Z_1, Z_2$ are the atomic numbers of projectile and target and $m, v$ are mass and velocity of the projectile. The condition $a/\lambda \gg 1$ is then

$$\eta \equiv \frac{a}{\lambda} = \frac{Z_1 Z_2 e^2}{\hbar v} \gg 1 \quad (8-7.2)$$

Furthermore, the collision must be nonadiabatic; otherwise no transition occurs. This means that the collision time $a/v$ must be short compared with the
nuclear periods $\tau$ to be excited, where

$$\tau = \frac{1}{\omega} = \frac{\hbar}{\Delta E} \quad (8.7.3)$$

and $\Delta E$ is the excitation energy of the target. If we indicate by $E$ the kinetic energy of the projectile, the condition of nonadiabatic collision gives

$$\xi = \frac{a}{v} \frac{\Delta E}{\hbar} = \frac{Z_1 Z_2 e^2}{h v} \frac{\Delta E}{2E} \ll 1 \quad (8.7.4)$$

When both conditions [Eqs. (8.7.2) and (8.7.4)] are satisfied, calculation gives for the total cross section for Coulomb excitation the approximate result for $\Delta E / E \ll 1$:

$$\sigma_{El} = \left( \frac{Z_1 e}{h v} \right)^2 a^{-2l+2} B (El, J_i \rightarrow J_f) f_{El} (\xi) \quad (8.7.5)$$

and

$$\sigma_{Ml} = \left( \frac{Z_1 e}{hc} \right)^2 a^{-2l+2} B (Ml, J_i \rightarrow J_f) f_{Ml} (\xi) \quad (8.7.6)$$

Note that the transition is characterized by its multipolarity and that the matrix element in $B$ is the same as the one entering into the spontaneous emission formula (8.3.11). The function $f(\xi)$ takes into account details of the orbit and is tabulated. For

$$\xi \ll 1 \quad f_{E2} = 1 \quad (8.7.6)$$

One of the most important cases of Coulomb excitation is that of $E2$ excitation of rotational levels $0^+, 2^+, 4^+$, etc., in even-even nuclei with

$$\Delta J = 2 \quad (8.7.7)$$

The matrix elements derived from experiment are sometimes many times larger than can be accounted for by single-particle models and are due to collective motions. They are connected with the intrinsic quadrupole moment $Q_0$ of such nuclei; e.g., for a $0^+ \rightarrow 2^+$ transition,

$$B (E2) = \frac{5}{16\pi} Q_0^2 \quad (8.7.8)$$

We see here a connection through the intrinsic quadrupole moment between the moment of inertia, the transition probability, and the observable electric quadrupole moment (cf. Sec. 6.12).
The intensities of $E2 \ 0^+ \rightarrow 2^+$ rotational transitions can be calculated from the static nuclear quadrupole moments. In some heavy nuclei, notably $^{238}$U and $^{230}$Th, one finds that the intensity of the $E4 \ 0^+ \rightarrow 4^+$ transition is detectable. This indicates a 16-pole permanent moment of the ground state. This moment is of the order of $1 \cdot eb^4$. It shows up also in alpha decay where it accounts for the low hindrance of some $\Delta J = 4$ transitions and in $\alpha$ inelastic scattering at about 40 MeV.

8.8 NUCLEAR FLUORESCENCE

In atomic physics, resonance and fluorescence radiation are important and easily observed phenomena. We would also expect to observe resonance radiation from nuclei; however, many early attempts to detect it failed. The reason for this is that the nuclear absorption lines are very narrow and thus absorb very little radiation from a continuum. On the other hand, if one tries to excite resonance radiation by the corresponding emission light as in the optical case, the recoil of the emitting nucleus shifts the light out of resonance with the absorber. Let us first consider a strictly monochromatic line. If the initial excited state has an energy $E$ above the ground states in the laboratory system, we have, by the principles of conservation of energy and momentum,

$$\hbar \omega + \frac{p^2}{2m} = E \quad (8-8.1)$$

with

$$p = \hbar \omega / c \quad (8-8.2)$$

where $\omega$ is the frequency of the light emitted and $p$ and $m$ are the recoil momentum and the mass of the nucleus supposed initially at rest. These relations give approximately

$$\hbar \omega = E \left(1 - \frac{E}{2mc^2}\right) \quad (8-8.3)$$

On the other hand, for absorption we need

$$\hbar \omega = E \left(1 + \frac{E}{2mc^2}\right) \quad (8-8.4)$$

to conserve energy and momentum. We see that, to have resonance, we must multiply the frequency of emission by

$$\sim \left(1 + \frac{E}{mc^2}\right) \quad (8-8.5)$$
or
\[ 1 + \frac{\hbar \omega}{mc^2} \]  \hspace{1cm} (8-8.6)

to find the necessary absorption frequency.

However, spectral lines have a natural width $\delta \omega$ associated with the mean life $\tau$ of the excited state by
\[ \delta \omega = \frac{1}{\tau} \]  \hspace{1cm} (8-8.7)

If this width is large compared to $\hbar \omega^2/mc^2$, the emission line will overlap the absorption line sufficiently to produce resonance radiation. This is the usual case in visible light. If, however, the natural width is insufficient to produce the desired overlap, as is the case in nuclear gamma rays, one can still modify the frequency, as seen by the absorber, with the help of the Doppler effect. If the source moves toward the absorber with velocity
\[ v = \frac{\hbar \omega}{mc} \]  \hspace{1cm} (8-8.8)

the Doppler shift compensates the recoil effects and resonance is obtained. For instance, in the case of $^{198}\text{Hg}$, $\hbar \omega = 0.41$ MeV, and $v$ must be equal to $0.67 \times 10^5$ cm sec$^{-1}$, which is attainable by mechanical means or by thermal agitation in a hot vapor. Actually nuclear resonance radiation has been observed (Moon, 1951) by using a source of radioactive gold 198 in rapid motion with respect to a mercury resonator. Recoil from a previous nuclear decay or reaction may also impart the necessary velocity to the nucleus. The study of the intensity of the resonance radiation as a function of velocity of the source can be made to yield information on $\tau$, which, in the case of $^{198}\text{Hg}$, is of the order of $10^{-11}$ sec.

Resonance absorption has been demonstrated in $^{191}\text{Ir}$ by Mössbauer (1958) by a different system. He used a $^{191}\text{Os}$ source and $^{191}\text{Ir}$ as the absorber, both cooled to a low temperature. The source decays by beta emission to an excited state at 129 keV of the stable $^{191}\text{Ir}$. The half-life of this state is $1.3 \times 10^{-10}$ sec and its natural width $(\delta \omega/\omega)_{\text{nat}} = 4 \times 10^{-11}$. The recoil energy of the free nucleus would be 0.047 eV, corresponding to a Doppler shift $(\Delta \omega/\omega)_{\text{Doppler}} = 3.6 \times 10^{-7}$. However, when the nucleus is bound in a crystal, its motion may be crudely compared to that of an oscillator. If the recoil energy is insufficient to raise the oscillator from its ground state to the first excited state, no energy can be transferred to the crystal lattice, and we have recoilless emission as if in Eqs. (8-8.3 and 4) the mass were infinite and emission and absorption frequency coincided.

To understand some of the features of the Mössbauer effect, consider an oversimplified model of a nucleus of mass $M$ bound in a harmonic oscillator
potential of frequency $\nu = \omega / 2\pi$. The hamiltonian of the system is

$$H = H_{\text{nucl}} + \frac{p^2}{2M} + \frac{M \omega^2}{2} x^2$$  \hspace{1cm} (8-8.9)

The eigenvalues are then

$$E_{a,n} = \epsilon_a + \hbar \omega (n + \frac{1}{2})$$  \hspace{1cm} (8-8.10)

where $\epsilon_a$ are the nuclear energy levels and $n$ is an integral number. The eigenfunctions are $\psi_a(r) u_n(x)$ where the first factor is nuclear and the second refers to the harmonic oscillator. The energy of the gamma emitted in the transition $a \rightarrow b$ is

$$E_\gamma = \epsilon_a - \epsilon_b + \hbar \omega (n' - n'')$$  \hspace{1cm} (8-8.11)

On the emission of the gamma ray the oscillator momentum changes abruptly, because of recoil, by the amount

$$q = E_\gamma / c \equiv (\epsilon_a - \epsilon_b) / c$$  \hspace{1cm} (8-8.12)

Assume initially the oscillator is in its ground state, eigenfunction $u_0(x)$, and write the transform of this eigenfunction in momentum space:

$$v_0(p) = (2\pi \hbar)^{-1/2} \int u_0(x) \exp(ipx / \hbar) \, dx$$  \hspace{1cm} (8-8.13)

The recoil at the gamma emission changes all momenta in the oscillator by $q$ and the eigenfunctions after the emission is, in momentum space, $v_0(p - q)$; from this function in momentum space we obtain the function in coordinate space immediately after the emission:

$$u'(x) = (2\pi \hbar)^{-1/2} \int v_0(p - q) \exp(-ipx / \hbar) \, dp = u_0(x) \exp(-iqx / \hbar)$$  \hspace{1cm} (8-8.14)

and the probability of finding the oscillator in the ground state is, according to the principles of quantum mechanics (Fe 61, pp. 40–43),

$$P_{00} = \left| \int u_0(x) u^*_0(x) \exp(iqx / \hbar) \, dx \right|^2$$  \hspace{1cm} (8-8.15)

where the subscripts on $P$ denote $n'$ and $n''$. 
Upon inserting the harmonic oscillator wave function in Eq. (8-8.15), one obtains

$$P_{00} = \exp\left( -\frac{q^2}{2M\hbar \omega} \right)$$  \hspace{1cm} (8-8.16)

One can calculate $P_{nn}$ or $P_{nm}$ similarly.

Equation (8-8.16) shows that recoilless emission is likely only if the recoil energy $q^2/2M$ is small compared with $\hbar \omega$, the energy corresponding to the excitation jumps in the oscillator.

One may describe a crystal as an ensemble of oscillators of the same frequency $\omega_E$, as Einstein did in his theory of specific heat.

**Figure 8-15** Relative probability for a gamma-ray transition simultaneous with the excitation of 1, 2, ..., $n$ oscillators in the crystal lattice for the Einstein model. The figures refer to two values of the ratio between the average recoil energy $\Delta E = E_0^2/2Me^2$ and $\hbar \omega_E$, $T = 0^o$K. [After R. L. Mössbauer, *Les Prix Nobel en 1961*, Stockholm, 1962.]
For $T \neq 0^\circ K$ the probability of finding an oscillator in the $n$th energy state is given by the Boltzmann factor $\exp(-\beta E_n)/\Sigma_n \exp(-\beta E_n)$ with $\beta = 1/kT$ and the probability of recoilless emission is

$$f = \sum_n \exp(-\beta E_n)P_{nn}/\sum_n \exp(-\beta E_n)$$

(8-8.17)

Figure 8-15 shows the probability of finding $n$ oscillators excited after the transition for $T = 0^\circ K$.

The actual spectrum of the lattice vibrations is much more complicated than in the Einstein schematization and as a better approximation one uses a Debye spectrum, replacing $\hbar \omega_E$ by $k\Theta$, where $\Theta$ is the Debye temperature. The development of these ideas in a quantitative form gives a formula for the probability $f$ of recoilless emission:

$$f = \exp\left\{ -\frac{3}{2} \frac{\hbar^2 \omega^2/2mc^2}{k\Theta} \left[ 1 + 4\left( \frac{T}{\Theta} \right)^2 \int_0^{\Theta/T} \frac{x \, dx}{e^x - 1} \right] \right\}$$

$$\approx \exp\left\{ -\frac{3}{2} \frac{\hbar^2 \omega^2/2mc^2}{k\Theta} \left[ 1 + \frac{2}{3} \left( \frac{\pi T}{\Theta} \right)^2 \right] \right\} \quad (T \ll \Theta) \quad (8-8.18)$$

The first term is independent of the temperature and shows that, even at absolute zero, the fraction of recoilless decays is large only if the recoil energy of the free nucleus is small compared with $k\Theta$. The probability of recoilless decay decreases with increasing temperature and it is negligible for temperatures large compared with $\Theta$ (Fig. 8-16). Similar effects occur in the absorption process.

![Graph](image)

**Figure 8-16** Fraction of recoilless transitions in iron or rhenium as a function of the temperature. [R. L. Mössbauer. Ann. Rev. Nucl. Sci., 12, 123 (1962).]
The experiments usually involve a source and an absorber of the same substance and the measurement of the amount of radiation absorbed. If one moves the absorber with respect to the source, the frequency of the absorption line changes by an amount

$$\frac{\Delta \omega}{\omega} = \nu / c$$  \hspace{1cm} (8-8.19)

and less radiation is absorbed, because the overlap of the emission and absorption curves is less complete (Figs. 8-17 and 8-18). It is thus possible to analyze the structure, or form, of a line by using "Doppler spectrometry."

Figure 8-17 Experimental arrangement: $A$, cryostat of absorber; $S$, rotating cryostat with source; $D$, scintillation detector; $M$, region in which the source is seen from $D$. [R. L. Mössbauer, Naturwiss., 45, 538 (1958).]

A quantitative illustration of the form of an absorption line as a function of the temperature of the lattice is given in Fig. 8-19. For an emission line one would have the same figure, but it would be reflected on the spike of the line, corresponding to the frequency of the recoilless transition.

The absorption of the radiation is naturally accompanied by the emission of fluorescent radiation from the absorbing nucleus and by the related emission of conversion electrons, etc. One of the examples of recoilless radiation most often studied is a line of 14.4-keV energy emitted by $^{57}$Fe in an $M1$ transition. It has a mean life of $10^{-7}$ sec and hence

$$\left( \frac{\Delta \omega}{\omega} \right)_{\text{nat}} \sim 3 \times 10^{-13}$$  \hspace{1cm} (8-8.20)
Figure 8-18 Fluorescent absorption in $^{191}$Ir as a function of the relative velocity between source and absorber. The upper scale on the abscissa shows the Doppler energy that corresponds to the velocity on the lower scale. $T = 88^\circ$K. [R. L. Mössbauer, Naturwiss., 45, 538 (1958).]

Figure 8-19 The absorption cross section per nucleus in a crystal of natural iridium, for a monochromatic gamma-ray beam and phonons having a Debye spectrum. The arrows give the cross section at zero relative velocity for the temperatures indicated. [W. M. Visscher, Ann. of Phys. 9, 194 (1960).]
The Doppler effect corresponding to this width is achieved with a velocity of about 0.01 cm sec\(^{-1}\) (Fig. 8-20); still sharper is a 13.3 keV line in \(^{73}\text{Ge}\).

\[
\begin{align*}
\text{JP} & \quad \text{JP} \\
\frac{3}{2} - & \quad \frac{57}{2} \text{Co} \quad 270 \text{d} \\
& \quad K \text{ capture} \\
& \quad 0.136 \\
& \quad 10\% \\
& \quad 90\% \\
\frac{3}{2} - & \quad 0.0144 \\
& \quad 1 \times 10^{-7} \text{ sec} \\
\frac{1}{2} - & \quad 0.0 \\
& \quad \text{\textsuperscript{57}Fe}
\end{align*}
\]

**Figure 8-20** Decay scheme of \(^{57}\text{Fe}\). The data to the left of the levels are their spins and parities, those to the right their energies. The times shown are the half-lives.

Such an unprecedented degree of monochromaticity has already permitted the observation of the nuclear Zeeman effect (Fig. 8-21) and it is an important tool in several fields of physics (e.g., gravitational red shift and solid-state investigations; Fig. 8-22).

An elegant application of Mössbauer spectroscopy is found in the so-called isomeric effect. Nuclei in an excited state have a slightly different radius than in the ground state. The electronic cloud surrounding the nucleus produces an energy shift, with respect to the energy for a point nucleus, given by

\[
\Delta W = (2/5)\pi Ze^2 r_n^2 |\psi(0)|^2
\]

where \(r_n\) is the nuclear radius. If the Mössbauer emitter and absorber are immersed in different materials, giving \(\psi_e(0)\) for the emitter and \(\psi_a(0)\) for the absorber, and if the nuclear radii for the excited and ground states are \(r_n^*\) and \(r_n\), respectively, one finds a difference between the energy of the line absorbed and emitted, and can be measured by the Doppler shift. (See Fig. 8-23.) The shift is

\[
\Delta E = (2/5)\pi Ze^2 (r_n^* - r_n^2)(|\psi_a(0)|^2 - |\psi_e(0)|^2)
\]
Figure 8-21 Fluorescent absorption in $^{57}$Fe showing hyperfine structure. Plotted is the observed relative transmission as a function of the velocity between source and absorber. (a) Randomly oriented magnetic field in source and absorber. (b) Source and absorber fields aligned parallel to one another. (c) Source and absorber fields aligned perpendicular to one another. [S. S. Hanna, J. Heberle, C. Littlejohn, G. J. Perlow, R. S. Preston, and D. H. Vincent, Phys. Rev. Letters, 4, 177 (1960).]
Figure 8.22 Absorption spectra for metallic $^{57}$Fe between room temperature and the Curie temperature. The single-line source ($^{57}$Co in Cu) was at the temperature of liquid nitrogen. [R. S. Preston, S. S. Hanna, and J. Heberle, *Phys. Rev.*, 128, 2207 (1962).]
Figure 8.23 Mössbauer absorption spectra of the 77-keV gamma rays of $^{197}$Au, measured at 4.2°K with sources of dilute impurities of $^{197}$Pt in V, Ru, W, and Pt, using a single-line gold metal absorber. [G. Kaindl and D. Salomon LBL 1666 see also Phys. Rev. B, 8, 1912 (1973).]
8. GAMMA EMISSION

BIBLIOGRAPHY

Ajzenberg-Selove, F. (AS 60).
Blatt, J. M., and V. Weisskopf (BW 52).
Jackson, J. D. (Ja 75).
Preston, M. A. (P 62).
Siegahn, K. (Si 65).

PROBLEMS

8-1 Calculate from Eq. (8-1.5) the mean life of a nuclear excited state that may decay by electric dipole radiation. Develop a numerical example and compare it with atomic radiation.

8-2 Draw the figures corresponding to Fig. 8-4 for the electric quadrupole indicated in Fig. 8-3.

8-3 An almost spherical surface defined by \( R(\theta) = R_0[1 + \beta P_2(\cos \theta)] \) has inside it a total charge \( Q \) uniformly distributed. The small parameter \( \beta \) varies harmonically in time with frequency \( \omega \). Keeping only lowest-order terms in \( \beta \) and making the long-wavelength approximation, calculate the nonvanishing multipole moments, the angular distribution of radiation, and the power radiated [From (Ja 75); compare Sec. 6-12 for collective nuclear motions.]
8-4 Suppose that part of the electromagnetic radiation emitted by a nucleus arises from an intrinsic magnetization:

\[ M(r', t) = M_0(r')e^{-i\omega t} + \text{complex conjugate} \]

(a) Obtain the general formula for the total radiation rate from such a system
(b) Show that the electric dipole contribution from this magnetization is zero.

8-5 Let us assume that we can crudely represent a nucleus as two oscillating currents in opposite directions, as shown in the figure below.

\[ \begin{align*}
I &= I_2 \sin \omega t \\
I_1 \sin \omega t &= I
\end{align*} \]

\[ R \quad \quad R \]

(a) If \( I_1 = 2(\omega / c) \), \( I_2 = \omega / c \) determine the multipole nature of the radiation, and obtain expressions for the angular distribution and the total decay rate. (b) Do the same for \( I_1 = I_2 = \omega / c \). (c) In part (b) let \( R = 5 \times 10^{-13} \) cm, \( \hbar \omega = 1 \) MeV, and determine the mean life of the excited state of the nucleus.

8-6 \(^{69}\)Ni has an excited state that decays to a lower excited state and then to the ground state in two successive electric quadrupole transitions. On the basis of this information, what are the possible spin and parity assignments for these two excited states of the Ni nucleus?

8-7 The lithium nucleus \(^7\)Li emits a 0.48-MeV gamma ray in a transition that goes from an excited state with angular momentum \( \frac{1}{2} \), odd parity, to the ground state with angular momentum \( \frac{3}{2} \), odd parity. (a) What are the possible choices for the multipolarity and nature of the emitted radiation? (b) Of these possibilities which one is likely to be the principal contributor to the transition? (c) Estimate the lifetime of the excited state.

8-8 Using the formulas

\[
\left( \frac{8\pi}{3} \right)^{1/2} Y_1^1 Y_{l-1}^m = \left[ \frac{(l + m)(l + m + 1)}{(2l + 1)(2l + 3)} \right]^{1/2} Y_{l+1}^m - \left[ \frac{(l - m)(l - m + 1)}{(2l + 1)(2l - 1)} \right]^{1/2} Y_{l-1}^m
\]

\[
\left( \frac{4\pi}{3} \right)^{1/2} Y_0^0 Y_l^m = \left[ \frac{(l + 1)^2 - m^2}{(2l + 1)(2l + 3)} \right]^{1/2} Y_{l+1}^m + \left[ \frac{l^2 - m^2}{(2l + 1)(2l - 1)} \right]^{1/2} Y_{l-1}^m
\]

\[
\left( \frac{8\pi}{3} \right)^{1/2} Y_1^{-1} Y_{l+1}^m = \left[ \frac{(l - m)(l - m + 1)}{(2l + 1)(2l + 3)} \right]^{1/2} Y_{l+1}^m - \left[ \frac{(l + m)(l + m + 1)}{(2l + 1)(2l - 1)} \right]^{1/2} Y_{l-1}^m
\]
calculate explicitly the integral (8-2.6) entering E1 radiation. Prove the formulas given above from the definition of $Y_{n}^{m}(\theta, \varphi)$ and properties of $P_{n}(x)$ as given in (Sc 68).

8-9 Discuss the conditions of energy and quantum numbers under which a nucleus can decay by electron-positron emission.

8-10 $^{89}$Y has an excited level 0.915 MeV above the ground state; it decays to the ground state with a half-life of about 16 sec. The initial state has spin $\frac{9}{2}$, the final state spin $\frac{1}{2}$, and there is a parity change in the transition. (a) What is the lowest multipole order which can contribute? Calculate the expected rate and compare it with the experimental result. (b) Suppose that we wanted to check whether, in the ground state, there was any mixture of other angular momenta than $\frac{1}{2}$ (i.e., check angular momentum conservation). Set a rough upper limit to the amplitude of an angular momentum = $\frac{3}{2}$ component in the final state, using as experimental data only the measured lifetime and transition energy. Assume strict parity conservation.

8-11 Plan an experiment to measure the internal conversion coefficient for the lines of $^{80}$Br.

8-12 In a mu-mesic atom the transition from the 2p to the 1s mesic orbit may occur by radiation or by an Auger process, in which an electron of the atom is emitted. Estimate the ratio of the probability of the second process with respect to the first one. Note the analogy to internal conversion.

8-13 Draw a diagram illustrating why the correlation between two $\gamma$ emissions can depend only on even powers of $\cos \theta$.

8-14 Three states having spins $J_{1}$, $J_{2}$, $J_{3}$ are connected by two gamma rays $\gamma_{1}$, $\gamma_{2}$ having an electric or magnetic multipolarity $l_{1}$, $l_{2}$. Show (or make a plausible argument) that the angular correlation between them does not depend on the energy of $\gamma_{1}$ and $\gamma_{2}$ or on their electric or magnetic character. Moreover, the cascade $J_{1} \rightarrow J_{2} \rightarrow J_{3}$ gives the same correlation as the cascade $J_{3} \rightarrow J_{2} \rightarrow J_{1}$. If $J_{2} = 0$, there is no correlation.

8-15 Using protons or alpha particles, one produces Coulomb excitation in a thin target. Adjust the velocities of the projectiles so that they have the same $\xi$ in both cases. What is the ratio of the cross sections and how does it depend on the electric multipole order?

8-16 Consider the resonance absorption of the 0.014-MeV gamma ray from $^{57}$Fe. (a) If there were no Mössbauer effect, and resonance absorption were to be observed by moving the source with respect to the absorber, how large would the relative speed have to be? How well controlled would it have to be (that is, what speed variations would destroy the effect?) (b) Considering the Mössbauer effect, suppose that the excited state has a magnetic moment of 0.5 nuclear magneton, the ground state has none, and the nucleus is in a field of $10^{5}$ G. For what values of the relative velocity of emitter and absorber will absorption peaks be observed? (c) Would you expect the 0.136-MeV gamma ray from the higher state of $^{57}$Fe to give rise to recoil-free emission? If not, why not?

8-17 Prove Eqs. (8-8.21) and (8-8.22).

8-18 Prove Eq. (8-8.16).