Light created when a ruby laser is focused in a cell of benzene emerges to form a pattern of brightly colored rings. New frequencies come from Raman resonances in the benzene. Courtesy of R. W. Terhune.

Laser beam enters a crystal of ammonium dihydrogen phosphate as red light and emerges as blue—the second harmonic. Courtesy of R. W. Terhune.
7.2 Design a resonator with \( R_1 = 20 \text{ cm}, \ R_2 = -32 \text{ cm}, \ l = 16 \text{ cm}, \ \lambda = 10^{-4} \text{ cm} \). Determine (a) the minimum spot size \( \omega_0 \); (b) its location; (c) the spot size \( \omega_m \) at the mirrors; and (d) the ratios of \( \omega_0, \omega_1, \) and \( \omega_2 \) to their respective confocal \( (R_1 = -R_2 = l) \) values.

7.3 Consider a confocal resonator with \( l = 16 \text{ cm}, \ \lambda = 10^{-4} \text{ cm} \), and reflectivities \( R_1 = R_2 = 0.995 \). Using Figure 7.7, choose the mirror’s aperture for which the total losses of the first high-order mode \( (TE_{01}) \) exceed 3 percent. For this choice of aperture, what is the loss of the fundamental mode? How can the choice of apertures be used to quench the oscillation of high-order transverse modes?

7.4 Show why the confinement diagram, Figure 7.4, is a graphic representation of the condition \( 0 < [1 - (1/R_1)][1 - (1/R_2)] < 1 \). Locate the eight resonators of Figure 7.1 on this diagram.

7.5 According to Figure 7.4 (or Equation 7.2-2), we can get stable modes with \( |R_d| = R_1, R_2 < 0 \), that is, an alternating sequence of equally converging and diverging lenses. Explain on physical grounds why this leads to net focusing.

HINT: consider the distance from the axis at which a ray traverses both types of lenses.

7.6 Using the ABCD law to derive the mode-characteristics (minimum spot size \( \omega_0 \) and the mirror spot sizes \( \omega_m \)) of a symmetric resonator with a mirror separation \( l \) and a radius of curvature \( R \).

HINT: show that the radius of curvature (of the self-consistent beam solution) of the phase front at the mirror positions is equal to that of the mirrors.

7.7 Show that an optical resonator formed by “replacing” any two phase fronts of a propagating Gaussian beam by reflectors (i.e., placing reflectors at \( z_1 \) and \( z_2 \) with radii of curvature equal to \( R(z_1) \) and \( R(z_2) \), respectively) is stable.

7.8 Obtain the mode confinement condition of an optical resonator formed by two identical mirrors with radii of curvature \( R \) and a separation \( l \) and a thin lens \( f \) at its center.

7.9 Show that the self-consistent beam parameter \( q \) (7.2-5) leads to beam radii of curvature at the mirrors’ positions which are identical to those of the respective mirrors i.e., \( R(z_1) = R_1, R(z_2) = R_2 \).

7.10 Show that after one round trip, a perturbation \( \Delta(1/q) \) of the complex beam parameter (from its steady state value (7.2-5)) becomes \( \delta(1/q) = e^{i\omega t} \Delta(1/q) \) where \( \cos \theta = (A + D) \cdot \Delta(1/q) \) is thus neutrally stable, \( |\delta(1/q)| = |\Delta(1/q)| \), in confined beams satisfying (7.2-6).

8.0 Introduction

In this chapter we consider the laws governing the interaction between atomic systems and electromagnetic radiation and ponder some of their consequences. We will make heavy use of the density matrix formalism introduced in Chapter 3. Some of the main topics considered include atomic susceptibilities, induced transitions, spontaneous transitions, amplification by an inverted atomic population, broadening mechanisms, and gain saturation.

We will keep the discussion general and not limit it to a specific atomic system so that the results may be widely applied. Some of the energy levels and transitions involved in individual laser materials will be considered in Chapter 10.

8.1 Density Matrix Derivation of the Atomic Susceptibility

In this section we will apply the density matrix formalism developed in Sections 3.14 and 3.15 to derive an expression for the susceptibility of an ensemble of atoms (or spins, ions, etc.) interacting with a time-harmonic electromagnetic field. The assumption is made that only two levels, with energies \( E_1 \) and \( E_2 \), are involved in the interaction as shown in Figure 8.1. This assumption is justified when the angular frequency \( \omega \) of the field satisfies \( \omega \sim (E_2 - E_1)/\hbar \). As a result the density matrix (3.14-6) is reduced to a \( 2 \times 2 \) matrix with elements \( \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22} \).

We assume that the interaction Hamiltonian \( H'(t) \) is of the dipole type and
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\[ \mathcal{H} = -\mu \mathbf{E}(t) \]  
\[ (8.1-1) \]

where \( \mu \) is the component of the dipole operator along the direction of field \( \mathbf{E}(t) \). In our initial analysis field \( \mathbf{E}(t) \) will be considered as a classical variable. The diagonal matrix elements of \( \mathcal{H}' \) are taken as zero

\[ \mu_{11} = \mu_{22} = 0 \]  
\[ (8.1-2) \]

as appropriate to transitions between states of definite parity. The phases of the eigenfunctions \( \{ \psi \} \) are taken, without loss of generality, such that

\[ \mu_{11} = \mu_{22} = \mu \]  
\[ (8.1-3) \]

The total Hamiltonian of the two-level system is

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}' \]  
\[ (8.1-4) \]

where \( \mathcal{H}_0 \) is the Hamiltonian of the system in the absence of any field.

Our task consists of solving for the ensemble average \( \langle \mu \rangle \) of the dipole moment of the atom that is induced by field \( \mathbf{E}(t) \). The value of \( \langle \mu \rangle \) is given according to (3.14-8) by

\[ \langle \mu \rangle = \text{tr}(\rho \mu) = \rho_{11}\mu_{11} + \rho_{12}\mu_{12} + \rho_{21}\mu_{21} + \rho_{22}\mu_{22} \]

and using (8.1-2)

\[ \langle \mu \rangle = \mu(\rho_{12} + \rho_{21}) \]  
\[ (8.1-5) \]

We represent the density matrix operator in terms of the eigenfunctions \( \psi_n \) of the unperturbed Hamiltonian \( \mathcal{H}_0 \) so that \( \mathcal{H}_0\psi_n = E_n\psi_n \).

Using (3.16-5) we obtain

\[ \frac{d\rho_{12}}{dt} = -\frac{i}{\hbar} [\{\mathcal{H}_0 + \mathcal{H}'\}, \rho_{12}] = -\frac{i}{\hbar} (\mathcal{H}_0\rho_{12} + E_2\rho_{12} - E_1\rho_{12} - \rho_{22}\mathcal{H}_0) \]

and after using (8.1-1) and defining the resonance frequency \( \omega_0 = (E_2 - E_1)/\hbar \)

\[ \frac{d\rho_{12}}{dt} = -i\omega_0\rho_{12} + i\frac{\hbar}{\hbar} E(t)(\rho_{12} - \rho_{21}) \]  
\[ (8.1-6) \]

In a similar manner we obtain

\[ \frac{d\rho_{21}}{dt} = -i\frac{\hbar}{\hbar} E(t)(\rho_{21} - \rho_{12}) \]

and

\[ \frac{d}{dt} (\rho_{11} - \rho_{22}) = 2i\frac{\hbar}{\hbar} E(t)(\rho_{12} - \rho_{21}) \]  
\[ (8.1-7) \]

where the last equation follows from the normalization condition \( \rho_{11} + \rho_{22} = 1 \).

The Inclusion of Collision Terms

If we pause to retrace the steps leading to Equations 8.1-5, 8.1-6, and 8.1-7 we recognize that the density matrix method of obtaining \( \langle \mu \rangle \) is formally equivalent to the conventional procedure whereby \( \langle \mu(t) \rangle = \int \psi^*\mu\psi \, dv \) (see Equation 1.1-16). Indeed, there is no particular merit to the use of the density matrix unless we take advantage of the fact that, according to Equation 3.15-2, it is defined as an ensemble average. First consider Equation 8.1-6. When perturbation field \( E(t) \) is turned off we would expect from (3.15-2) that \( \rho_{12} \) would decrease and eventually approach zero as the relative phase coherence among the \( S \) eigenfunctions in the ensemble is lost via "collisions." These collisions are characterized by the fact that they conserve the average energy (or level occupation) but cause a loss of (ensemble) information involving the phase \( \phi_n \) in the wavefunction

\[ \psi_n(r, t) = u_n(r)e^{-i(E_n\hbar + \phi_n) t} \]

Such collisions were considered first in magnetic resonance (Ref. 1) and are referred to as the "spin-spin" relaxation time \( T_1 \) (Ref. 2). In our case they will be considered in Chapter 10 in connection with pressure broadening in molecular lasers in which this type of collision determines the absorption linewidth.

We will incorporate the loss of phase coherence into the density matrix
formalism by modifying (8.1-6) to

$$\frac{dp_{12}}{dt} = -i\omega_0 p_{12} + \frac{i}{\hbar} (\rho_{11} - \rho_{22}) E(t) - \frac{\rho_{12}}{T_2}$$

(8.1-8)

Using (3.14-6) it follows that $p_{12}$ is the probability of finding an atom in the $i$th state. If $N$ is the density of atoms, $N(\rho_{11} - \rho_{22}) = \Delta N$ becomes the (average) density of the population difference between the two levels. Let the equilibrium $\{E(t) = 0\}$ value of $\rho_{11} - \rho_{22}$ be denoted by $(\rho_{11} - \rho_{22})_0$, and let us assume that when $E(t)$ is turned off, the population difference $\Delta N$ relaxes toward its equilibrium value $N(\rho_{11} - \rho_{22})_0$ with a time constant $T_2$. We may consequently rewrite (8.1-7) as

$$\frac{d}{dt} (\rho_{11} - \rho_{22}) = \frac{2i\mu E(t)}{\hbar} (\rho_{11} - \rho_{22}) - \frac{(\rho_{11} - \rho_{22})_0}{\tau}$$

(8.1-9)

Next consider the special case when the local perturbing field $E(t)$ is time harmonic so that

$$E(t) = E_0 \cos \omega t$$

(8.1-10)

In addition we see from (8.1-8) that the nondriven [i.e., $E(t) = 0$] behavior of $p_{12}$ is $p_{12}(t) = p_{12}(0) e^{i\omega t - \sigma_{12}(t)}$ so that, for $\omega = \omega_0$, it is useful to define new "slowly" varying variables $\sigma_{12}$ and $\sigma_{21}$ through the relations

$$\rho_{12}(t) = \sigma_{12}(t) e^{-i\omega t}$$

$$\rho_{12}(t) = \sigma_{21}(t) e^{i\omega t} = \rho_{12}^*$$

(8.1-11)

Using (8.1-10) and (8.1-11) we rewrite equations (8.1-8) and (8.1-9) as

$$\frac{d\sigma_{12}}{dt} = i(\omega - \omega_0)\sigma_{12} + \frac{i\mu E_0}{\hbar} (\rho_{11} - \rho_{22}) - \frac{\sigma_{12}}{T_2}$$

(8.1-12)

$$\frac{d}{dt} (\rho_{11} - \rho_{22}) = \frac{i\mu E_0}{\hbar} (\sigma_{12} - \sigma_{21}) - \frac{(\rho_{11} - \rho_{22})_0}{\tau}$$

(8.1-13)

In deriving (8.1-12) we kept only terms with $\exp(-i\omega t)$ time dependence while in (8.1-13) we kept only the terms with no exponential time dependence, thus ignoring factors with time dependence $\exp(2i\omega t)$ and $\exp(-2i\omega t)$. This neglect of the nonsynchronous terms is physically justified since their contribution averages out to zero in times that are short compared to those of interest (but long compared to $2\pi/\omega$).

1. This is not necessarily the thermal equilibrium value since some "pump" mechanism may be present that causes $\Delta N$ at equilibrium to have some fixed value that is different from its thermal equilibrium value.

2. Since an inelastic collision also causes a loss of phase coherence, in cases where $\tau < T$, we use $\tau$ instead of $T_2$. This point is discussed in Section 8.6.
We define a normalized lineshape function \( g(\nu) \) by
\[
g(\nu) = \frac{2T_z}{1 + 4\pi^2(\nu - \nu_0)^2T_z^2} = \frac{\Delta \nu/2\pi}{(\nu - \nu_0)^2 + (\Delta \nu/2)^2}
\] (8.1-20)
with a full width at half maximum \( \Delta \nu = (\pi T_z)^{-1} \).

We note that \( \chi''(\nu) \), which according to (5.1-19) is proportional to the absorption, and \( \chi' \) are in the form of
\[
\chi''(\nu) \propto \Delta N g(\nu) \\
\chi'(\nu) \propto \Delta N (\nu_0 - \nu) g(\nu)
\] (8.1-21)
We will refer to \( g(\nu) \) as given by (8.1-20) as the normalized Lorentzian Lineshape function. The normalization constant was chosen so that
\[
\int_{-\infty}^{\infty} g(\nu) \, d\nu = 1
\] (8.1-22)

The derivation leading to (8.1-19) shows that the Lorentzian lineshape is characteristic of collision \((T_z, T_z')\) dominated transitions. A plot of the Lorentzian absorption \( \chi'' \) and dispersion \( \chi' \) in the limit \( 4\Omega^2 T_z T_z' \ll 1 \) is shown in Figure 8.2.

### Saturation

One consequence of (8.1-17), (8.1-19), and (8.1-21) is that the population difference \( \Delta N \) as well as \( \chi' \) and \( \chi'' \) decrease with increasing field intensity. This phenomenon, which is called saturation, becomes noticeable when \( 4\Omega^2 T_z T_z' > 1 + (\omega - \omega_0)^2 T_z^2 \), or, using \( \Omega = \mu E_{\text{rms}}/2h \), when
\[
\frac{\mu^2 E_{\text{rms}}^2 T_z}{h} > 1 + (\omega - \omega_0)^2 T_z^2
\] (8.1-23)

Another consequence of saturation is a broadening of the Lorentzian lineshape function from a zero field value of \( \Delta \nu = (\pi T_z)^{-1} \) to
\[
\Delta \nu_{\text{sat}} = \Delta \nu \sqrt{1 + \frac{\mu^2 E_{\text{rms}}^2 T_z}{h}}
\] (8.1-24)

We will return to this topic in Section 8.7 in connection with gain saturation.

### The Kramers-Kronig Relations

According to a fundamental theorem of the theory of complex variables, the real and imaginary parts of a complex function \( f(z) \) that has no poles in the lower (or upper) \( z \) plane are related by the Hilbert transformation (Ref. 3). When applied to the complex susceptibility function \( \chi(\omega) = \chi'(\omega) - i\chi''(\omega) \), these transformations for the case of \( \chi(\infty) = 0 \) are
\[
\chi'(\omega) = -\frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} \, d\omega' \\
\chi''(\omega) = -\frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega} \, d\omega'
\] (8.1-25)
where \( \text{P.V.} \) stands for the Cauchy principal value of the integral that follows. The equations in (8.1-25) are derived in Appendix 1. In the present context they are known as the Kramers-Kronig relations (Ref. 4).

We next inquire whether the solutions for \( \chi'(\omega) \) and \( \chi''(\omega) \), (8.1-19), satisfy (8.1-25). To do this we must find out if \( \chi(\omega) \) is analytic in the lower half of the \( \omega \) plane. From (8.1-19) we have
\[
\chi(\omega) = \chi'(\omega) - i\chi''(\omega) = \frac{\mu^2 \Delta N_0}{\epsilon_0 \hbar} \frac{\omega - [\omega_0 - (i/T_z)(1 + \epsilon)]}{[\omega - [\omega_0 + (i/T_z)(1 + \epsilon)]\epsilon]}
\] (8.1-26)
where \( \epsilon = \mu^2 E_{\text{rms}}^2 T_z / \hbar^2 \). In the absence of saturation, \( \epsilon = 0 \), \( \chi(\omega) \) has a single pole at \( \omega = \omega_0 + i/T_z \). For this case \( \chi'(\omega) \) and \( \chi''(\omega) \) obey (8.1-25). The actual demonstration is left as an exercise. For \( \epsilon \neq 0 \), \( \chi(\omega) \) has poles at \( \omega = \omega_0 \pm i/T_z(1 + \epsilon) \) and the Kramers-Kronig relations do not apply. The significance of the presence or absence of poles from the point of view of the transient behavior of the system is discussed in Appendix 1.
Connection with Magnetic Resonance

The treatment leading to (8.1-19) is formally identical to the Bloch equations solutions of magnetic resonance (Ref. 1). In the simple case of $m_J = \pm \hbar$ transitions we associate \( J = 1 \) with \( \text{II} \) and \( J = 0 \) with \( \text{II} \). To complete the analogy we use 

\[ (8.1-2) \]

\[ \text{II} \Rightarrow \text{II}, \quad (\text{II}) \Rightarrow (\text{II}) \]

where \( H(t) \) is the amplitude of the linearly polarized rf magnetic field, \( \hbar \) is the Bohr magneton, \( J \) is the total angular momentum operator, \( M_0 \) is the equilibrium magnetization, \( T \) is the "spin-lattice" relaxation time, \( \gamma \) is the magnetogyric ratio, and \( g \) is the Landé g factor.

With these substitutions and defining \( M(t) = \text{Re}[\chi(\omega)H(t)e^{\gamma t}] \) (8.16) and (8.17) yield

\[ (8.1-27) \]

\[ \chi'(\omega) = \frac{1}{1 + (\omega - \omega_0)^2 T_1^2 + \gamma^2 H^2 / T_1}, \]

\[ \chi''(\omega) = \frac{\gamma T_2}{1 + (\omega - \omega_0)^2 T_1^2 + \gamma^2 H^2 / T_1}, \]

that are the same as the conventional solutions of Bloch equations (Ref. 2).

8.2 The Significance of \( \chi(\nu) \)

According to (5.1-3) the electric displacement vector is defined by

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} + \mathbf{P}_{\text{transition}} = \varepsilon \mathbf{E} + \varepsilon_0 \chi \mathbf{E} \]

where the complex notation is used and the polarization is separated into a resonant component \( \mathbf{P}_{\text{transition}} \) due to the specific atomic transition and a nonresonant component \( \mathbf{P} \) that accounts for all the other contributions to the polarization. We can rewrite the last equation as

\[ (8.2-1) \]

\[ \mathbf{D} = \varepsilon \left[ 1 + \frac{\varepsilon_0}{\varepsilon} \chi(\omega) \right] \mathbf{E} = \varepsilon'(\omega) \mathbf{E} \]

so that the complex dielectric constant becomes

\[ (8.2-2) \]

\[ \varepsilon'(\omega) = \varepsilon \left[ 1 + \frac{\varepsilon_0}{\varepsilon} \chi(\omega) \right] \]

We have thus accounted for the effect of the atomic transition by modifying \( \varepsilon \) according to (8.2-2). Having derived \( \chi(\omega) \), using detailed atomic information as in Section 8.1, we can now ignore its physical origin and proceed to treat the wave propagation in the medium with \( \varepsilon' \) given by (8.2-2), using Maxwell's equations.

As an example of this point of view we consider the propagation of a plane electromagnetic wave in a medium with a dielectric constant \( \varepsilon'(\omega) \). According to (5.2-9), the wave has the form of

\[ (8.2-3) \]

\[ E(z, t) = \text{Re}[\varepsilon'(\omega) e^{i\omega t}] \]

where

\[ k' = \nu \sqrt{\mu \varepsilon} = k \left[ 1 - \frac{\varepsilon_0}{\varepsilon} \right] \]

and

\[ \nu = \frac{k' \sqrt{\mu \varepsilon}}{\nu} \]

Expressing \( \chi(\nu) \) in terms of its real and imaginary components, \( \chi = \chi' - i\chi'' \), leads to

\[ (8.2-4) \]

\[ k' = k \left[ 1 + \frac{\chi''(\nu) \nu^2}{n^2} \right] \]

where \( n = (\varepsilon / \varepsilon_0)^{1/2} \) is the index of refraction in the medium far from resonance. Substituting (8.2-4) back into (8.2-3), we find that in the presence of the atomic transition the wave propagates according to

\[ (8.2-5) \]

\[ E(z, t) = \text{Re}[\varepsilon'(\omega) e^{i\omega t}] \]

The result of the atomic polarization is thus to change the phase delay per unit length from \( k \) to \( k + \Delta k \), where

\[ (8.2-6) \]

\[ \Delta k = \frac{\chi''(\nu) \nu^2}{2n^2} \]

as well as to cause the amplitude to vary exponentially with distance according to \( e^{ixx} \), where

\[ (8.2-7) \]

\[ \chi(\nu) = - \frac{\chi''(\nu) \nu^2}{n^2} \]

It is quite instructive to rederive (8.2-7) using a different approach. According to (5.1-19), the average power absorbed per unit volume from an electromagnetic field with a y component is

\[ (8.2-8) \]

\[ \frac{\text{Power}}{\text{Volume}} = \frac{E_y(t) dP_y(t)}{dt} = \frac{1}{2} \text{Re}[E_y(\nu)(\nu)] \]

where \( E \) and \( P \) are the complex electric field and polarization amplitudes in the y direction, respectively, and horizontal bars denote time averaging. Using \( P = \varepsilon_0 \chi \mathbf{E} \) in (8.2-8), we obtain

\[ (8.2-9) \]

\[ \frac{\text{Power}}{\text{Volume}} = \frac{\omega \varepsilon_0}{2} \chi' \mathbf{E} \]

The absorption of energy at a rate given by (8.2-9) must lead to a change of the
wave intensity $I$, according to

$$I(z) = I e^{\gamma z}$$  \hspace{1cm} (8.2-10)$$

where

$$\gamma(v) = I^{-1} \frac{dl}{dz}$$  \hspace{1cm} (8.2-11)$$

Conservation of energy thus requires that

$$\frac{dl}{dz} = -(\text{power absorbed per unit volume}) = -\frac{\varepsilon_0 c}{2} \chi^* |E|^2$$

Using the last result in (8.2-11) as well as the relations

$$I = \frac{\varepsilon_0 c}{2n} |E|^2$$

where $c$ is the velocity of light in vacuum, gives

$$\gamma(v) = -\frac{k \chi^*(v)}{n^2}$$

in agreement with (8.2-7).

### 8.3 Spontaneous and Induced Transitions

In the formalism of Section 8.1 the electromagnetic field was taken as a classical variable. This is sufficient for treating the coherent interaction of atoms with strong optical fields. There are, however, several aspects of this problem that require the quantization of the electromagnetic field as well as that of the atomic variables. The most notable of these is the phenomenon of spontaneous emission. This term is used to indicate the transition from an excited atomic state, say level 2 in Figure 8.1 to a lower level (e.g., 1) in the absence of any externally applied inducing field. This transition is accompanied by the emission of a photon of energy $E_2 - E_1$.

We start by considering an atom excited initially to level 2 that is placed inside a large optical enclosure. We will calculate first the rate for the process indicated in Figure 8.3 in which the atom undergoes a transition from 2 to 1 due to its interaction with a single radiation mode, for example, $l$ (of the enclosure). The mode $l$, simultaneously, makes a transition from state $|n_l \rangle$ to $|n_l + 1 \rangle$. The interaction Hamiltonian is

$$H' = -e E(z, t) \cdot \mathbf{r} = -e E_0(z, t) y$$  \hspace{1cm} (8.3-1)$$

where the mode is assumed to be plane-wavelike with $E \parallel J$. The position of the atom is $z$.

Using the operator expression for $E_0$, Eq. (5.6-15), we rewrite (8.3-1) as

$$H' = i e y \sqrt{\frac{\hbar}{V_0}} (a_l^\dagger - a_l) \sin k z$$  \hspace{1cm} (8.3-2)$$

where $V$ is the volume of the enclosure.

The initial state is $|2, n_l \rangle$ in which the atom is in level 2 and mode $l$ has $n_l$ quanta while the final state is $|1, n_l + 1 \rangle$ in which the atom is in the lower state 1 while the field has gained a photon. The initial and final states are of nearly the same energy so that the transition rate is calculated using (3.12-20) as

$$\Gamma' = \frac{2 \pi e^2 \alpha^2}{V_0} \langle 1, n_l + 1 | a_l^\dagger | 2, n_l \rangle |y_{n_l}^2| \sin(kz) \delta(E_2 - E_1 - \hbar \omega)$$  \hspace{1cm} (8.3-3)$$

Using (2.2-27) we have $\langle n_l + 1 | a_l^\dagger | n_l \rangle = \sqrt{n_l + 1}$ so that the last equation can be written as

$$\Gamma' = \frac{2 \pi e^2 \alpha^2}{V_0} \langle n_l + 1 | \sin(kz) \delta(E_2 - E_1 - \hbar \omega)$$  \hspace{1cm} (8.3-4)$$

where $y_{n_l}^2 = |\langle 1 | y | 2 \rangle|^2$.

It follows that the rate of induced emission

$$W' = \frac{2 \pi e^2 \alpha \gamma_{12}}{V_0} n_l \sin k z \delta(E_2 - E_1 - \hbar \omega)$$  \hspace{1cm} (8.3-5)$$

into a single mode is equal to the spontaneous transition rate into the mode

$$W_{\text{spont}} = \frac{2 \pi e^2 \alpha \gamma_{12}}{V_0} \sin k z \delta(E_2 - E_1 - \hbar \omega)$$  \hspace{1cm} (8.3-6)$$
multiplied by the number of quanta \( n \) in the mode. A similar derivation shows that the induced absorption rate for a \( 1 \rightarrow 2 \) transition is equal to the induced emission rate.

**The Spontaneous Lifetime**

The lifetime due to spontaneous transitions into all of the continuum modes is called the spontaneous lifetime. To calculate it we first allow for the possibility that the lower state is \( g_i \)-fold degenerate and multiply (8.3-6) by \( g_i \). The factor \( \sin^2 k_z \) is replaced by its average value over a large number of modes that is \( 1/2 \). We next multiply by the number of modes per unit energy (5.7-3) \( \frac{E}{\hbar^3} \frac{\Delta \nu}{v_n} \) and integrate over all energies. The result is

\[
W_{\text{spont}} = \frac{1}{t_{\text{spont}}} = \frac{2n^2 \mu^2 \omega^3 g_i}{\hbar c^3},
\]

where \( \mu = e y_{12} \) and \( \hbar \omega = E_2 - E_1 \).

**The Induced Transition Rate Due to a Monochromatic Field**

The induced rate for a \( 2 \rightarrow 1 \) transition is given by (8.3-5) for the case of an atom interacting with a single mode. This result can be used to obtain the transition rate due to a traveling monochromatic wave at frequency \( v \). We start by recalling that expression (8.3-5) applies to the case where the transition energy is exactly \( E_2 - E_1 \). In general \( E_2 - E_1 \) is not known precisely, and the probability that \( E_2 - E_1 \) occur in interval \( E \rightarrow E + dE \) is given by \( g(E) dE = (1/\hbar)g(v) dE \) where \( g(v) \) is the normalized lineshape function for the \( 2 \rightarrow 1 \) transition. We thus multiply (8.3-5) by \( h^2 g(v) \) and integrate over all energies. We allow for a degeneracy \( g_i \) of level 1 and get

\[
(W_{ni}) = \frac{\pi e^2 y_{12}^2 \omega^3 g_i}{\hbar V_n} g(n) \quad (8.3-8)
\]

where we replaced \( \sin^2 k_z \) by a spatial average of \( 1/2 \) as appropriate for a traveling wave with the same energy density as that of the \( i \)th mode.

Finally we relate the mode excitation number \( n \) in (8.3-8) to the wave intensity \( I_n \), by

\[
I_n = cn \hbar n \frac{\nu}{n^2}
\]

and eliminate \( y_{12}^2 \) in (8.3-8) through the use of (8.3-7). This leads to

\[
(W_{ni}) = \frac{\lambda^2 I_n}{8 \pi \hbar n \nu^2 t_{\text{spont}}} g(n) \quad (8.3-9)
\]

which is the key result of this section. It points out the proportionality between the induced transition rate and the intensity \( I_n \) of the inducing field as well as the functional dependence on \( g(v) \), the lineshape function. In the case of a collision broadened transition such as that leading to (8.1-20), only frequencies within \( -\Delta \nu \) of line center are thus effective in inducing a transition. If the interacting atom is initially in the lower level 1, the relevant matrix element, which would appear in (8.3-3),

\[
\langle 2, n - 1 | y_{12} | 1, n \rangle = (2 | y | 1) \sqrt{n}
\]

and thus if \( n_i = 0 \) the transition rate is zero. It follows that no spontaneous transitions exist from a low level to a (energetically) higher level. In addition, the \( 2 \rightarrow 1 \) and \( 1 \rightarrow 2 \) induced transition rates are the same. We can thus use (8.3-9) to obtain

\[
(W_{i2}) = W_{i1} (\frac{g_2}{g_1}) = (g_2) \frac{\lambda^2 I_n}{8 \pi \hbar n \nu^2 t_{\text{spont}}} g(v) \quad (8.3-10)
\]

The factor \( g_2/g_1 \) accounts for the degeneracy \( g_i \) of level 2, since (8.3-9) was derived assuming in (8.3-7) a degeneracy \( g_i \) for level 1.

### 8.4 The Gain Coefficient

Consider the passage of a monochromatic wave frequency \( v \) through an assembly of atoms of the type shown in Figure 8.2. The atom density is \( N_2 \) (atoms/m\(^3\)) in level 2 and \( N_1 \) (atoms/m\(^3\)) in level 1. The excess of induced \( 2 \rightarrow 1 \) over \( 1 \rightarrow 2 \) transitions per unit volume per unit time gives rise to an induced power

\[
\frac{\text{Power}}{\text{Volume}} = [N_2(W_{i1}) - N_1(W_{i2})] h \nu \quad (8.4-1)
\]

where spontaneous transitions are ignored. Using (8.3-9) and (8.3-10) leads to

\[
\frac{\text{Power}}{\text{Volume}} = (N_2 - N_1) \frac{\lambda^2 g(v) I_n}{8 \pi \hbar n \nu^2 t_{\text{spont}}} \quad (8.4-2)
\]

Assuming that this power is added to the inducing wave it follows that the latter grows according to

\[
\frac{dI_n(z)}{dz} = (\text{power/volume}) = \gamma(v) I_n(z) \quad (8.4-3)
\]
where, using (8.4-2),
\[ \gamma(v) = \frac{(N_2 - N_1) \lambda^2}{8\pi n^2 t_{\text{spont}}} g(v) \] (8.4-4)

If \( N_2 \) and \( N_1 \) are independent of \( z \), a situation that prevails when the pumping is uniform and saturation effects are negligible, the wave intensity grows exponentially according to
\[ L(z) = L(0) e^{\gamma z} \] (8.4-5)

The problem of exponential amplification has also been treated in Section 8.2. It can be readily shown, using (8.1-17), (8.1-19), (8.1-20), and the equivalence
\[ g_2 \sim N_2, g_1 \sim N_1, g \sim N, g \]
that \( \gamma(v) \) as given by (8.2-7) is the same as that of (8.4-4).

Example—Gain in a Ruby Crystal. Consider an \( \text{Al}_2\text{O}_3 \) crystal with 0.5 wt. percent of \( \text{Cr}_2\text{O}_3 \) added. This is "pink" ruby. The crystal contains about \( 2.4 \times 10^{18} \) chromium atoms per cubic centimeter. This crystal is a common laser material and will be described in some detail in Chapter 10.

We assume that some pump agency causes an inversion
\[ N_1 \gg N_2 \]
Using the following data:
\[ t_{\text{spont}} = 3 \times 10^{-3} \text{ sec} \]
\[ \lambda = 0.6943 \text{ \mu m} \]
\[ n = 1.77 \]
\[ \Delta \nu = 1/g(\nu_0) = 2 \times 10^{11} \text{ Hz at 300K} \]
in (8.4-4) leads to a gain constant
\[ \gamma(\nu_0) \sim 5 \times 10^{-3} \text{ cm}^{-1} \]

8.5 The Einstein Treatment of Induced and Spontaneous Transitions

The results of Section 8.3 can also be derived using classical arguments as was done originally by Einstein (Ref. 5).

Consider the interaction of an assembly of identical atoms with a radiation field whose energy density is distributed uniformly in frequency in the vicinity of the transition frequency. Let the energy density per unit frequency be \( \rho(v) \).

We assume that the induced transition rates per atom from \( 2 \rightarrow 1 \) and \( 1 \rightarrow 2 \) are both proportional to \( \rho(v) \) and take them as
\[ \begin{align*}
(W_{12})_1 &= B_{12} \rho(v) \\
(W_{21})_1 &= B_{21} \rho(v)
\end{align*} \] (8.5-1)

where \( B_{12} \) and \( B_{21} \) are constants to be determined. The total downward (\( 2 \rightarrow 1 \)) transition rate is the sum of the induced and spontaneous contributions
\[ W_{21} = B_{21} \rho(v) + A \] (8.5-2)

The spontaneous rate \( A \) was discussed in Section 8.3. The total upward (\( 1 \rightarrow 2 \)) transition rate is
\[ W_{12} = (W_{12})_1 = B_{12} \rho(v) \] (8.5-3)

Our first task is to obtain an expression for \( B_{12} \) and \( B_{21} \). Since the magnitude of the coefficients \( B_{12} \) and \( B_{21} \) depends on the atoms and not on the radiation field, we consider, without loss of generality, the case where the atoms are in thermal equilibrium with a blackbody (thermal) radiation field at temperature \( T \). In this case the radiation density is given by (5.7-4) as
\[ \rho(v) = \frac{8\pi n^2 \hbar^2 v^2}{c^3} \left( \frac{1}{e^{\hbar v/kT} - 1} \right) \] (8.5-4)

Since at thermal equilibrium the average populations of levels 2 and 1 are constant with time, it follows that the number of \( 2 \rightarrow 1 \) transitions in a given time interval is equal to the number of \( 1 \rightarrow 2 \) transitions; that is,
\[ N_2 W_{21} = N_1 W_{12} \] (8.5-5)

where \( N_1 \) and \( N_2 \) are the population densities of level 1 and 2, respectively. Using (8.5-2) and (8.5-3) in (8.5-5), we obtain
\[ N_1[B_{12} \rho(v) + A] = N_2 B_{21} \rho(v) \]
and, substituting for \( \rho(v) \) from (8.5-4),
\[ N_1 \left[ B_{12} \frac{8\pi n^2 \hbar^2 v^2}{c^3(e^{\hbar v/kT} - 1)} + A \right] = N_2 \left[ B_{21} \frac{8\pi n^2 \hbar^2 v^2}{c^3(e^{\hbar v/kT} - 1)} \right] \] (8.5-6)

Since the atoms are in thermal equilibrium, the ratio \( N_2/N_1 \) is given by the Boltzmann factor
\[ \frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\hbar v/kT} \] (8.5-7)

Equating \( N_2/N_1 \) as given by (8.5-6) to (8.5-7) gives
\[ \frac{8\pi n^2 \hbar^2 v^2}{c^3(e^{\hbar v/kT} - 1)} = \frac{A(g_2/g_1)}{B_{12} e^{\hbar v/kT} - B_{21} (g_2/g_1)} \] (8.5-8)
The last equality can be satisfied only when

$$B_{21} = B_{21} \frac{g_2}{g_1}$$  \hspace{1cm} (8.5-9) $$

and, simultaneously,

$$A = \frac{8\pi n^3 h\nu^3}{c^4}$$  \hspace{1cm} (8.5-10) $$

We can, using (8.5-10), rewrite the induced transition rate (8.5-1) as

$$\langle W_{21} \rangle = \frac{A c^3}{8\pi n^3 h\nu^3} \rho(v) = \frac{c^3}{8\pi n^3 h\nu^3} \int g(v) \rho(v) dv$$  \hspace{1cm} (8.5-11) $$

Equation 8.5-11 gives the transition rate per atom due to a field with a uniform (white) spectrum with energy density per unit frequency $\rho(v)$. In quantum electronics our main concern is in the transition rates that are induced by a monochromatic (i.e., single-frequency) field of frequency $\nu$. Let us denote this transition rate as $\langle W_{21} \rangle$. We have established in Section 8.3 that the strength of interaction of a monochromatic field of frequency $\nu$ with an atomic transition is proportional to the lineshape function $g(v)$, so $\langle W_{21} \rangle \propto g(v)$. Furthermore, we would expect $\langle W_{21} \rangle$ to go over into $\langle W_{12} \rangle$, as given by (8.5-11) if the spectral width of the radiation field is gradually increased from zero to a point at which it becomes large compared to the transition linewidth. These two requirements are satisfied if we take $\langle W_{21} \rangle$ as

$$\langle W_{21} \rangle = \frac{c^3 \rho_0}{8\pi n^3 h\nu^3} \int g(v) dv$$  \hspace{1cm} (8.5-12) $$

where $\rho_0$ is the energy density (joules per cubic meter) of the electromagnetic field inducing the transitions. To show that (8.5-12) indeed goes over smoothly into (8.5-11) as the spectrum of the field broadens, we may consider the broad spectrum field as made up of a large number of closely spaced monochromatic components at $\nu_k$ with random phases and then by adding the individual transition rates obtained from (8.5-12) obtain

$$\langle W_{21} \rangle = \sum_{\nu_k} \langle W_{21} \rangle(\nu_k) = \frac{c^3}{8\pi n^3 h\nu^3} \int \rho_0 g(v) dv$$  \hspace{1cm} (8.5-13) $$

where $\rho_0$ is the energy density of the field component oscillating at $\nu_k$. We can replace the summation of (8.5-13) by an integral if we replace $\rho_0$ by $\rho(v) dv$ where $\rho(v)$ is the energy density per unit frequency; thus, (8.5-13) becomes

$$\langle W_{21} \rangle = \frac{c^3}{8\pi n^3 h\nu^3} \int \rho(v) g(v) dv$$  \hspace{1cm} (8.5-14) $$

In situations where $\rho(v)$ is sufficiently broad compared with $g(v)$, and thus the variation of $\rho(v)/\nu^3$ over the region of interest (where $g(v)$ is appreciable) can be neglected, we can pull $\rho(v)/\nu^3$ outside the integral sign, obtaining

$$\langle W_{21} \rangle = \frac{c^3}{8\pi n^3 h\nu^3} \int g(v) dv$$  \hspace{1cm} (8.5-15) $$

where we used the normalization condition

$$\int g(v) dv = 1$$

This agrees with (8.5-11).

Returning to our central result, (8.5-12), we can rewrite it in terms of the intensity $I = \rho_0 / h$ (watts per square meter) of the optical wave as

$$\langle W_{21} \rangle = \frac{A c^3 I}{8\pi n^3 h\nu^3} \int g(v) dv = \frac{\lambda^3 I}{8\pi n^3 h\nu^3} \int g(v) dv$$  \hspace{1cm} (8.5-15) $$

where $c$ is the velocity of propagation of light in vacuum and $t_{\text{spont}} = 1/\lambda$. This is the same result as that given by (8.3-9).

### 8.6 Homogeneous and Inhomogeneous Broadening

The term broadening is used to denote the finite spectral width of the response of atomic systems to electromagnetic fields. The broadening may manifest itself, as an example, in a plot of the absorption as a function of frequency or in the frequency dependence of the gain of a laser medium. Such plots are included in Chapter 10.

We distinguish between two main classes of broadening mechanisms:

**Homogeneous Broadening** (Ref. 6). In this case the atoms are indistinguishable and have the same transition energy $E_2 - E_1$. The broadening is due to one or a combination of the following factors: (1) inelastic collisions with phonons or with other atoms (or molecules); (2) transitions to other levels—these may be spontaneous radiative transitions or nonradiative; (3) elastic phase-destroying collisions; and (4) broadening due to the interaction with an electromagnetic field (power broadening).

The case of homogeneous broadening is the one considered in Section 8.1 where it was found to give rise to a Lorentzian response curve of the form

$$\chi''(\omega) = \frac{1}{1 + (\omega - \omega_0)^2 T_2^2 + \frac{E_2^2}{h^2} T_2^2 \tau}$$  \hspace{1cm} (8.6-1) $$

with a power dependent width

$$\Delta \nu_{\text{eff}} = \Delta \nu \sqrt{1 + \frac{\mu^2 E_2^2 T_2^2 \tau}{h^2}}$$  \hspace{1cm} (8.6-2) $$
where $\Delta \nu = (\pi T)^{-1}$. $T$ is the time constant characterizing the loss of atomic coherence as in (8.1-8) and is thus given by

$$\frac{1}{T} = \sum \frac{1}{\tau_i} \quad (8.6-3)$$

where the summation is over all the processes (collisions, transitions) that interrupt the coherent field-atom interaction.

**Inhomogeneous Broadening (Ref. 6).** In this case the atoms are distinguishable, and the broadening reflects a spread in the individual resonant (transition) energies of the atoms. Two main examples of this type of broadening are that of impurity ions in a host crystal and of molecules in low pressure gases.

In the first case the energy levels, hence the transition frequencies, depend on the immediate crystalline surrounding of each atom. The ever-present random strain, as well as other types of crystal imperfections, cause the crystal surroundings to vary from one ion to the next, thus effecting a spread in the transition frequencies.

In the second example, the transition frequency $\nu$ of a gaseous atom (or molecule) is Doppler-shifted due to the finite velocity of the atom according to

$$\nu = \nu_0 + \frac{v_x}{c} v_0 \quad (8.6-4)$$

where $v_x$ is the component of the velocity along the direction connecting the observer with the moving atom, $c$ is the velocity of light, and $\nu_0$ is the frequency corresponding to a stationary atom. The Maxwell velocity distribution function of a gas with atomic mass $M$ that is at equilibrium at temperature $T$ is

$$f(v_x, v_y, v_z) = \frac{M}{2\pi kT} \exp \left[ -\frac{M}{2kT} (v_x^2 + v_y^2 + v_z^2) \right] \quad (8.6-5)$$

$f(v_x, v_y, v_z) dv_x dv_y dv_z$ corresponds to the fraction of all the atoms whose $x$ component of velocity is contained in the interval $v_x$, $v_x + dv_x$, while simultaneously, their $y$ and $z$ components lie between $v_y$ and $v_y + dv_y$, and $v_z$ and $v_z + dv_z$, respectively. Alternatively, we may view $f(v_x, v_y, v_z) dv_x dv_y dv_z$ as the a priori probability that the velocity vector $v$ of any given atom terminates within the differential volume $dv_x dv_y dv_z$ centered on $v$ in velocity space so that

$$\int \int \int f(v_x, v_y, v_z) dv_x dv_y dv_z = 1 \quad (8.6-6)$$

According to (8.6-4) the probability $g(v) dv$ that the transition frequency is between $\nu$ and $\nu + dv$ is equal to the probability that $v_x$ will be found between $v_x = (\nu - \nu_0)c/v_0$ and $\nu + dv - \nu_0)c/v_0$ irrespective of the values of $v_y$ and $v_z$ [since if $v_x = (\nu - \nu_0)c/v_0$, the Doppler-shifted frequency will be equal to $\nu$ regardless of $v_y$ and $v_z$]. This probability is thus obtained by substituting

$$g(v) dv = \left( \frac{M}{2\pi kT} \right)^{3/2} \exp \left[ -\frac{M}{2kT} (v_x^2 + v_y^2 + v_z^2) \right] \quad (8.6-7)$$

Using the definite integral

$$\int_0^{\nu_0} e^{-M/M} M^{1/2} dv_x dv_y dv_z = \frac{(2\pi kT)^{3/2}}{M^{1/2}} \quad (8.6-8)$$

we obtain, from (8.6-7),

$$g(v) = \frac{c}{\nu_0} \left( \frac{M}{2\pi kT} \right)^{1/2} e^{-M/M} M^{1/2} dv_x dv_y dv_z \quad (8.6-8)$$

for the normalized Doppler-broadened lineshape. The functional dependence of $g(v)$ in (8.6-8) is referred to as Gaussian. The width of $g(v)$ in this case is taken as the frequency separation between the points where $g(v)$ is down to half its peak value. It is obtained from (8.6-8) as

$$\Delta \nu_D = \frac{c}{\nu_0} \left( \frac{M}{2\pi kT} \right)^{1/2} \frac{1}{\sqrt{\ln 2}} \quad (8.6-9)$$

where the subscript $D$ stands for "Doppler." We can reexpress $g(v)$ in terms of $\Delta \nu_D$ as

$$g(v) = \frac{2(\ln 2)^{1/2}}{\pi^{1/2} \Delta \nu_D} e^{-M/M} M^{1/2} \quad (8.6-10)$$

### 8.7 Gain Saturation in Systems with Homogeneous and Inhomogeneous Broadening

The most important difference between atomic systems with homogeneous and inhomogeneous broadening manifests itself in their power saturation. Specifically, when such systems are used as laser media their gain decreases with increasing field intensity. The amount of decrease and its spectral dependence are different in these two cases and will be considered below.

**Homogeneous Broadening.** Gain saturation here is due to the decrease of the population inversion with field intensity. The gain is given by (8.4-4) as

$$\gamma(v) = \frac{\Delta N}{8\pi n \Delta \nu_D} g(v) \quad (8.7-1)$$

where $g(v)$ is the normalized lineshape function

$$g(v) = \frac{2T_i}{1 + 4\pi^2 (v - v_0)^2 T_i^2} \quad (8.7-2)$$
as derived in Section 8.1. The population inversion density $\Delta N = N_2 - N_1 (g_2/g_1)$ is given by (8.1-17) as

$$\Delta N = \Delta N_0 \frac{1 + (\omega - \omega_0)^2 T_1^2}{1 + (\omega - \omega_0)^2 T_1^2 + 4\Omega^2 T_1 \tau}$$

(8.7-3)

where $\Omega^2 = (\mu E_0/2 \hbar)^2$. We substitute (8.7-2) and (8.7-3) in (8.7-1). In the resulting expression we eliminate the matrix element $\mu = ey_{02}$ through the use of (8.3-7). The result is

$$\gamma(v) = \frac{\Delta N_0 \lambda^2 g(v)}{8 \pi n^2 t_{\text{prompt}}} \left( \frac{1}{1 + L(v)/I(v)} \right)$$

where $\gamma(v)$ is the unsaturated ($E_0 = 0$) gain, $I$ is the intensity (watts/m$^2$) given by

$$I = \frac{c n e I_0^2}{2}$$

and $I(v)$ the intensity at which the gain, at $v$, is reduced by a factor of 1/2 (compared to the zero intensity case) is called the "saturation intensity" and is given by

$$I(v) = \frac{4 \pi n^2 h v}{(\tau t_{\text{prompt}})^2} \frac{g'(v)}{g(v)}$$

(8.7-5)

The degeneracy factor $g$ does not appear here since according to (8.3-7) it is included in $t_{\text{prompt}}$. The inversion lifetime $\tau$ as defined by (8.1-9) is equal in most cases to the actual lifetime (i.e., not necessarily the radiative lifetime) of the upper laser level.

Returning to (8.7-5) we note that $I(v)$, the saturation intensity is inversely proportional to $g(v)$ so that saturation becomes increasingly difficult off line-center.

**Example—Gain Saturation in a Ruby Laser.** We use the ruby example of Section 8.4. We take $\tau = t_{\text{prompt}}$, $1/g(v) = 2 \times 10^4$ Hz at 300 K. Using these data in (8.7-5) gives

$$I(v) \sim 467 \text{ watts/cm}^2$$

**Inhomogeneous Broadening.** In the first part of this section we considered the reduction in optical gain—that is, saturation—due to the optical field in a homogeneous laser medium. In what follows we treat the problem of gain saturation in inhomogeneous systems.

According to the above discussion, in an inhomogeneous atomic system the individual atoms are distinguishable, with each atom having a unique transition frequency $(E_2 - E_1)/\hbar$. We can thus imagine the inhomogeneous medium as made up of classes of atoms each designated by a center frequency $v_i$. Furthermore, we define a function $p(v_i)$ so that the $a \text{ priori}$ probability that an atom has its center frequency between $v_i$ and $v_i + dv_i$ is $p(v_i) dv_i$. 

$$\int_{-\infty}^{\infty} p(v_i) dv_i = 1$$

(8.7-6)

since any atom has a unit probability of having its $v_i$ between $-\infty$ and $\infty$.

The atoms within a given $v_i$ are considered as homogeneously broadened, having a lineshape function $g^i(v)$ that is normalized so that

$$\int_{-\infty}^{\infty} g^i(v) dv = 1$$

(8.7-7)

We can define the transition lineshape $g(v)$ by taking $g(v) dv$ to represent the $a \text{ priori}$ probability that a spontaneous emission will result in a photon whose frequency is between $v$ and $v + dv$. Using this definition we obtain

$$g(v) dv = \left[ \int_{-\infty}^{\infty} p(v_i) g^i(v) dv_i \right] dv$$

(8.7-8)

which is a statement of the fact that the probability of emitting a photon of frequency between $v$ and $v + dv$ is equal to the probability $g^i(v) dv$ of this occurrence, given that the atom belongs to class $\xi$, summed up over all the classes.

If the total unsaturated inversion is $\Delta N_0$ (atoms/m$^3$) then the inversion due to atoms in $dv_i$ is $\Delta N_0 p(v_i) dv_i$ and the contribution of that class $\xi$ alone to the exponential gain constant at $v$ is given by (8.7-4) as

$$\gamma(v) = \frac{\Delta N_0 \lambda^2}{8 \pi n^2 t_{\text{prompt}}} \left[ \int_{-\infty}^{\infty} \frac{p(v_i) dv_i}{g^i(v)} \right]$$

(8.7-9)

where $\phi = \tau t_{\text{prompt}}$. It follows from the definition of the gain constant (8.4-3) that the contribution of the various classes $v_i$ to $\gamma(v)$ are additive so that

$$\gamma(v) = \frac{\Delta N_0 \lambda^2}{8 \pi n^2 t_{\text{prompt}}} \int_{-\infty}^{\infty} \frac{p(v_i) dv_i}{g^i(v)}$$

(8.7-10)

This is our basic result.

As a first check on (8.7-8) we shall consider the case in which $I \ll 4 \pi n^2 h \phi \lambda^2 g^i(v)$ and therefore the effects of saturation can be ignored. Using (8.7-8) in (8.7-10)

$$\gamma(v) = \frac{\Delta N_0 \lambda^2}{8 \pi n^2 t_{\text{prompt}}} g(v)$$

which is the same as (8.7-4) with $I = 0$. This shows that in the absence of
saturation the expressions for the gain of a homogeneous and an inhomogeneous atomic system are identical.

Our main interest in this treatment is in deriving the saturated gain constant for an inhomogeneously broadened atomic transition. If we assume that in each class all the atoms are identical (homogeneous broadening), we can use \( g'(v) \),

\[ g'(v) = \frac{\Delta v}{2\pi[(\Delta v)^2 + (v - \nu)^2]} \]  

(8.7-11)

where \( \Delta v \) is called the homogeneous linewidth of the inhomogeneous line. Atoms with transition frequencies that are clustered within \( \Delta v \) of each other can be considered as indistinguishable. The term “homogeneous packet” is often used to describe them. Using (8.7-11) in (8.7-10) leads to

\[ \gamma(v) = \frac{\Delta N N^2}{16\pi n^2 l_{\text{spont}}} \int_0^\pi \frac{p(\nu_0) d\nu_0}{(\nu - \nu_0)^2 + (\Delta v/2)^2 + (\Delta v L/v_{\text{sp}})^2} \]  

(8.7-12)

In the extreme inhomogeneous case, the width of \( p(v_0) \) is by definition very much larger than the remainder of the integrand in (8.7-12), and thus it is essentially a constant over the region in which the integrand peaks. In this case we can pull \( p(\nu_0)_{\nu_0} = p(\nu) \) outside the integral sign in (8.7-12) obtaining

\[ \gamma(v) = \frac{\Delta N N^2}{16\pi n^2 l_{\text{spont}}} p(\nu) \int_0^\pi \frac{d\nu_0}{(\nu - \nu_0)^2 + (\Delta v/2)^2 + (\Delta v L/v_{\text{sp}})^2} \]  

(8.7-13)

Using the definite integral

\[ \int_0^\pi \frac{dx}{x^2 + a^2} = \frac{\pi}{a} \]

to evaluate (8.7-13), we obtain

\[ \gamma(v) = \frac{\Delta N N^2 p(\nu)}{8\pi n^2 l_{\text{spont}}} \frac{1}{\sqrt{1 + (\Delta v L/2\pi n^2 \nu \Delta v)}} \]  

(8.7-14)

\[ = \gamma_0(v) \frac{1}{\sqrt{1 + (L/L_s)}} \]  

(8.7-15)

and

\[ L = \frac{2\pi n^2 \nu \Delta v}{\phi L_s} \]  

(8.7-16)

is the saturation intensity of the inhomogeneous line. A comparison of (8.7-15) and (8.7-16) to (8.7-4) and (8.7-5) reveals two essential differences between the saturation behavior of homogeneous and inhomogeneous systems.

1. The inhomogeneous system saturates more “slowly” as indicated by the square root in (8.7-15). This can be explained by the fact that although the inversion per class (packet) \( \nu_0 \) decreases as in (8.7-4), this is partly compensated by the fact that more classes are brought into the interaction as \( L \) increases in accordance with (8.6-2). If we multiply the form of (8.6-2) by (8.7-4), the result is the inverse square law dependence of (8.7-15).

2. The saturation intensity in the inhomogeneous case does not depend on the position in the lineshape. That is, \( L \) in (8.7-16) does not depend on \( g(v) \) as does the saturation intensity of the homogeneous case (8.7-5).

“Hole” Burning. To further appreciate the difference between the saturation behavior of homogeneous and inhomogeneous media consider the following case. A strong field at \( \nu \) is applied to the medium and simultaneously a very weak probing signal at \( \nu' \) is used to measure the gain \( \gamma(\nu') \). Our task is to determine the form of \( \gamma(\nu') \) for both homogeneous and inhomogeneous media.

Consider the homogeneous case first. The gain at \( \nu' \) is given according to (8.7-1) by

\[ \gamma(\nu') = \Delta N \frac{\lambda^2}{8\pi n^2 l_{\text{spont}}} g(\nu') \]  

where \( \Delta N \) is the inversion in the presence of the strong field at \( \nu \). \( \Delta N \) is given by (8.7-3) that, when used in the last equation, leads to

\[ \gamma(\nu') = \gamma_0(\nu') \left[ \frac{1 + 4\pi^2 (\nu - \nu_0)^2 T_1^2}{1 + 4\pi^2 (\nu - \nu_0)^2 T_1^2 + \frac{E_0^2}{8\pi^2 n^2 \nu_0^2 T_1 T_{\text{sp}}}} \right] \]  

(8.7-17)

where \( E_0 \) is the amplitude of the strong field at \( \nu \) and \( \gamma_0(\nu') \) is the nonsaturated \((E_0 = 0) \) gain function

\[ \gamma(\nu') = \Delta N \frac{\lambda^2}{8\pi n^2 l_{\text{spont}}} g(\nu') \]

The important conclusion is that \( \gamma(\nu') \) has the same frequency dependence as \( \gamma_0(\nu') \) but it is reduced in magnitude by the factor inside the square brackets of (8.7-17).

In the case of an inhomogeneously broadened gain medium the situation is more complicated. We start by using (8.7-11) to rewrite (8.7-10) as

\[ \gamma(\nu) = \frac{\Delta N N^2}{8\pi n^2 l_{\text{spont}}} \int_0^\pi d\nu_0 p(\nu_0) g'(\nu) \left[ \frac{(\Delta v)^2}{2} + (\nu - \nu_0)^2 \right] \]  

The integrand is clearly proportional to the contribution to the gain at \( \nu' \) due to the atomic packet centered on \( \nu_0 \). It follows directly that the quantity inside the square bracket represents the factor by which this contribution is reduced due to the saturating field at \( \nu \). The gain exercised by the weak probing signal at \( \nu' \) is
thus the unsaturated gain $\gamma'(v')$ multiplied by this local reduction factor, that is,

$$\gamma'(v') = \gamma_0'(v') \left[ \frac{\Delta v^2}{\Delta v_0^2} + (v - v')^2 \right]$$

The main features of (8.7-18) is that $\gamma'(v')$ is essentially identical to $\gamma_0'(v')$ except for frequencies $v'$ in the vicinity of the saturating frequency $v$. Here the gain is depressed over a frequency interval approximately equal to

$$\Delta v_{satur} = \Delta v \sqrt{1 + \frac{L}{I}}$$

and the gain at $v' = v$ is reduced by a factor $(1 + f_0/v)$. $I$, is the saturation intensity defined by (8.7-16). This depressed region is usually referred to as a "hole" and the phenomenon just described, as "hole burning."

The gain profile $\gamma'(v')$ of the probing signal is sketched in Figure 8.4 for both cases of broadening. A different approach to the problem of gain and gain saturation based on rate equations is given in Reference 7.

**REFERENCES**

3. See, for example, P. M. Morse and H. Feshback, Methods of Theoretical Physics (McGraw-Hill, New York, 1953), p. 372.

**PROBLEMS**

8.1 Show that relation (8.5-10)

$$A \frac{8\pi \nu^3 n^3}{c^4}$$

is consistent with Equation 8.3-4 according to which

$$\frac{W_{induced}}{W_{satur}} per mode = n$$

where $n =$ number of quanta in the mode.
8.2 Determine the peak absorption coefficient \( \alpha(\nu_0) \) due to a transition at
\( \nu_0 = 3 \times 10^{14} \text{ Hz} \) where \( N_i = 0, N_f = 10^{14} \text{ cm}^{-3} \), the full width of the Gaussian absorption curve is 400 cm\(^{-1} \), and \( \tau_{\text{spont}} = 10^{-8} \text{ sec} \). Defining the optical density as
\[
\log_{10} \frac{I_0}{I_{\text{out}}}
\]
where \( I \) denotes intensity, what is the optical density at \( \nu_0 \) for a 1-cm path length of material? At what temperature will the rate for the transition induced by blackbody radiation equal the spontaneous emission rate?

8.3 Calculate the classical lifetime \( \tau_{\text{class}} \) = energy/(radiated power) of an electron oscillating so that \( r = r_0 \cos(2\pi\nu t) \) where \( r \) is the electron position.

8.4 (a) Acquaint yourself with the concept of the oscillator strength of a transitions (Refs. 8, 9).
(b) What is the oscillator strength of the transition described in 8.2?
(c) Show that the oscillator strength \( f_{12} \) for a transition 1 \( \leftrightarrow \) 2 at a frequency \( \nu \) is equal to \( \tau_{\text{class}}/M_{\text{avg}} \).

8.5 Derive (8.1-15)
8.6 Show that \( \chi'(\omega) \) and \( \chi''(\omega) \) in (8.1-19) obey the Kramers-Kronig relations in the limit of negligible saturation (\( \Omega = 0 \)).

9.0 Introduction

Proposals for using stimulated emission from a system of inverted population for microwave amplification were made, independently, by Weber (Ref. 1), Gordon, Zeiger, and Townes (Ref. 2), and Basov and Prokhorov (Ref. 3). The first operation of such an amplifier was by the Columbia University group of Gordon, Zeiger, and Townes. This group is responsible for the name “maser,” an acronym for “microwave amplification by the stimulated emission of radiation.” The first maser utilized a microwave transition in the ammonia (NH\(_3\)) molecule. The feasibility of maser action at optical and near optical frequencies was considered in a paper by Schawlow and Townes (Ref. 4) in 1958. In 1960, less than two years later, Maiman (Ref. 5) succeeded in operating a pulsed ruby laser (acronym for “light amplification by stimulated emission of radiation”). The first continuous wave (cw) laser was a He-Ne gas laser announced in 1961 (Ref. 6). Laser action in semiconductors was demonstrated in 1962 and is described in Chapter 10.

In the last chapter we found that a medium with an inverted atomic population is capable of amplifying radiation at frequencies near that of the atomic transition. In this chapter we consider what happens if such an amplifying medium is placed within an optical resonator. The ever-present zero point fluctuation fields of the resonator modes will now be amplified. The few modes whose losses are sufficiently low (i.e., those with a high \( Q \)) may experience net amplification. These modes will grow in intensity until their gain saturates at a level equal to the loss and steady-state oscillation prevails.