Beams, Modes and Resonators'

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PARAXIAL RAYS

The passage of paraxial rays through linear optical structures can be described by ray transfer matrices. Paraxial rays are rays which have small slopes with respect to the optic axis. As indicated in Figure 14-1, a paraxial ray in a given cross-section \( z = \text{const} \) of the structure is characterized by its distance \( x \) from the optic \( (z) \) axis and by its angle \( x' \) with respect to that axis. The ray matrix, or "ABCD-matrix," relates the input quantities \( x \) and \( x' \) to the output quantities \( x_2 \) and \( x'_2 \) by

\[
\begin{pmatrix}
  x_2 \\
  x'_2
\end{pmatrix} =
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}
\]

The matrix elements generally satisfy the relation

\[ AD - BC = 1. \]

Fig. 14-1. Reference planes of an optical system. A typical ray path is indicated.

The matrix elements are related to the focal length \( f \) of the system and to the location of the principal planes by

\[
\begin{align*}
f &= \frac{1}{C} \\
 h_1 &= \frac{D - 1}{C} \\
 h_2 &= \frac{A - 1}{C}
\end{align*}
\]

* Most of the following material is based on the review article "Laser Beams and Resonators," which appeared jointly in Appl. Optics, 2, 1550, October, 1966 and in the Proc. IEEE, 54, 1312, October, 1966. This article should be consulted for further details, explanations, derivations and references.
where $h_1$ and $h_2$ are the distances of the principal planes from the input and output planes, as shown in Figure 14-1.

Table 14-1 lists the ray transfer matrices of six elementary optical structures. The matrix of No. 1 describes the ray transfer over a distance $d$. No. 2 describes the transfer of rays through a thin lens of focal length $f$. Here the input and output planes are immediately to the left and right of the lens. No. 3 is a combination of the first two. It governs rays passing first over a distance $d$ and then through a thin lens. If the sequence is reversed the diagonal elements are interchanged. The matrix of No. 4 describes the rays passing through two structures of the No. 3 type. The ray transfer matrix for a lenslike medium of length $d$ is given in No. 5. In this medium the refractive index varies quadratically with the distance $r$ from the optic axis.

$$n = n_0 - i \eta_2 r^2.$$  

The matrix of a dielectric material of index $n$ and length $d$ is given in No. 6. The matrix is referred to the surrounding medium of index 1.

**LASER BEAMS**

**Gaussian Beams**

Lasers behave, in many respects, similarly to uniform plane waves; however, their intensity distributions are not uniform, but are concentrated near the axis of propagation and their phase fronts are slightly curved. A laser oscillating in a fundamental transverse (TEM$_{00}$) mode produces a beam with a transverse amplitude distribution that is approximately Gaussian in every cross-section of the beam. This is illustrated in Figure 14-2.

**BEAM RADIUS.** The width of the beam is measured by the "beam radius" $w$. This is defined as the distance at which the field amplitude is 1/e times that on the axis.

**BEAM EXPANSION.** A Gaussian beam contracts to a minimum diameter $2w_0$ at the beam waist where the phase front is plane. The beam expands with the distance $z$ from the waist according to

$$w^2(z) = w_0^2 [1 + (2z/w_0)^2]$$

The resulting beam contour $w(z)$ is illustrated in Figure 14-3. The far field diffraction angle $\theta$ is given by

$$\theta = \lambda w_0 / z.$$  

The radius of curvature $R$ of the wavefront changes according to

$$R(p) = \sqrt{1 + (w_0^2 / (2z))^2}.$$  

As the beam propagates, $R$ varies from infinity at the waist to a minimum of $R_{\text{min}} = 2w_0^2 / \lambda$ at $z = w_0^2 / \lambda$ and approaches $z$ asymptotically. Also, the beam experiences a phase shift $\Phi$ relative to an ideal uniform plane wave; it is a phase advance and is given by

$$\Phi = \arctan \left( \frac{2\pi w_0}{\lambda z} \right).$$
COMPLEX BEAM PARAMETER. The real beam parameters $w$ and $R$ are used to define a complex beam parameter $q$

$$q = \frac{i}{2} \frac{\lambda}{\pi w^2} \left( \frac{1}{R} - \frac{1}{f} \right)$$

Using the $q$ parameter the laws of beam propagation simplify to

$$q_2 = q_1 + \gamma,$$

where $q_1$ and $q_2$ refer to the input and output planes respectively, and $\gamma$ is the distance between these planes.

EFFECT OF LENS. A thin lens of focal length $f$ transforms the $q$ parameters according to

$$q_2 = \frac{1}{q_1} \left( 1 - \frac{1}{f} \right) q_1,$$

where $q_1$ and $q_2$ are measured immediately before and after the lens.

ABCD-LAW. The input ($q_1$) and output ($q_2$) parameters of a laser beam passing through a general optical system characterized by its $\text{ABCD}$ matrix are related by

$$q_1 = \frac{A q_2 + B}{C q_2 + D}.$$

Mode Matching

For matching the modes of one optical structure to those of another, one often needs to transform a given Gaussian beam into another prescribed Gaussian beam. This transformation can be accomplished with a thin lens, as indicated in Figure 14-4. The focal length $f$ of this lens must be larger than a characteristic length $f_c$, defined by the confocal parameters $b_1$ and $b_2$ of the two beams as

$$f^2 = 4b_1 b_2.$$

The confocal parameters are related to the waist diameters $2w_1$ and $2w_2$ of the beams by

$$w_1 = f w_2 [b_2], \quad b_2 = 2n w_1 [f].$$

For matching, the distances $d_1$ and $d_2$ between the lens and the beam waists are adjusted to be

$$d_1 = f \pm \sqrt{b_1} \sqrt{(f_2^2 - f^2)^{1/2}} - 1,$$
$$d_2 = f \pm \sqrt{b_2} \sqrt{(f_2^2 - f^2)^{1/2}} - 1,$$

where one can choose to use either both plus signs or both minus signs.

Table 14-2 lists formulas for the parameters of the exit beams that correspond to the modes of various optical structures. They are the confocal parameter $b$ and the distance $t$, which gives the waist location of the emerging beam. System No. 1 is a resonator formed by a flat mirror and a spherical mirror of radius $R$. System No. 2 is a resonator formed by two equal spherical mirrors. System No. 3 is a resonator formed by mirrors of unequal curvature. System No. 4 is a resonator formed by two equal spherical mirrors, with the reflecting surfaces deposited on planoconcave optical plates of index $n$. These plates act as negative lenses and change the characteristics of the emerging beam. System No. 5 is a sequence of thin lenses of equal focal lengths $f$. System No. 6 is a system of two irises with equal apertures spaced at a distance $d$. Shown are the parameters of a beam that will pass through both irises with the least possible beam diameter. This is a beam which is "confocal" over the distance $d$. This beam will also pass through a tube of length $d$ with the optimum clearance. A similar situation is shown in System No. 7, which corresponds to a beam that is confocal over the length of optical material of index $n$. System No. 8 is a spherical mirror resonator filled with material of index $n$, or an optical material with curved end surfaces, where the beam passing through it is assumed to have phase fronts that coincide with these surfaces.

Circle Diagrams

The propagation of Gaussian laser beams can be represented graphically on circle diagrams or "beam charts." There one can follow a beam as it propagates in free space or passes through lenses. The charts contain the beam parameters $w$, $R$, $w_0$, and $z$ defined above in the section on Gaussian beams, and the confocal parameter $b = 2n w_0 f$. The chart proposed by Collins is plotted in the complex plane of

$$f = \frac{\lambda}{\pi w^2} + i \frac{1}{R},$$

and is shown in Figure 14-5. In this plane lines of constant $b/2 = \pi w_0 f$ and lines of constant $z$ appear as circles through the origin. A beam is represented by a circle of constant $b$, and the beam parameters $w$ and $R$ at a distance $z$ can be read off the coordinate axes. When the beam passes through a lens of focal length $f$, the phase front is changed and a new beam is formed. In the diagram this is represented by a vertical line of length $1/f$, which connects the circles corresponding to the input and output beams. The angle $\Phi$ shown in the figure is equal to the relative phase shift $\Phi$ experienced by the beam, as given above.

The chart proposed by Li is the dual to the Collins chart and plotted in the complex plane of

$$-\frac{b}{2} = \frac{\lambda}{\pi w^2} + i \frac{1}{f}.$$
### TABLE 14-2. FORMULAS FOR THE CONFOCAL PARAMETER AND THE LOCATION OF BEAM WAIST FOR VARIOUS OPTICAL STRUCTURES

<table>
<thead>
<tr>
<th>№</th>
<th>OPTICAL SYSTEM</th>
<th>( \frac{b}{n} = \frac{\pi d^2}{\lambda} )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{d}{R} )</td>
<td>( \sqrt{d(R-d)} )</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{d}{R} )</td>
<td>( \frac{1}{2} \sqrt{d(R-d)} )</td>
<td>( \frac{1}{2} d )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{d}{R_1} ) ( \frac{d}{R_2} )</td>
<td>( \frac{d}{R_1 + R_2 - 2d} ) ( \frac{d}{R_1 + R_2 - 2d} )</td>
<td>( \frac{ndR}{n' R_d(d^2 - d)} ) ( \frac{2R + d(n'-n)}{2R + d(n'-n)} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{d}{R} ) ( \frac{d}{R} )</td>
<td>( \sqrt{d(R-d)} )</td>
<td>( \frac{1}{2} d )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{d}{R} ) ( \frac{d}{R} )</td>
<td>( \sqrt{d^2(R-d)} )</td>
<td>( \frac{1}{2} d )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{d}{N} ) ( \frac{d}{N} )</td>
<td>( \frac{1}{2} d )</td>
<td>( \frac{1}{2} d )</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{d}{N} ) ( \frac{d}{N} )</td>
<td>( \frac{d}{2R} )</td>
<td>( \frac{2R}{2R} )</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{d}{R} ) ( \frac{d}{n} ) ( \frac{d}{d} ) ( \frac{d}{R} ) ( \frac{d}{R} )</td>
<td>( \frac{ndR}{n' R_d(d^2 - d)} ) ( \frac{dR}{2\sqrt{R_d(d^2 - d)}} )</td>
<td>( \frac{dR}{ndR(d^2 - d)} ) ( \frac{dR}{2\sqrt{R_d(d^2 - d)}} )</td>
</tr>
</tbody>
</table>

Fig. 14-5. Geometry for the \( \theta \)-plane circle diagram.

**Gaussian Beam Chart**

Fig. 14-6.
Other circle diagrams include those proposed by Gordon which allow the graphic determination of resonator parameters (see below for definitions). Table 14-3 gives a comparison of the parameters that appear in the various circle diagrams.

<table>
<thead>
<tr>
<th>TABLE 14-3. PARAMETERS OF BEAM CHARTS</th>
<th>Collins</th>
<th>Li</th>
<th>Gordon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter plotted as ordinate</td>
<td>I/R</td>
<td>z</td>
<td>a₁</td>
</tr>
<tr>
<td>Diameter of circles centered on ordinate</td>
<td>R</td>
<td>1/5s</td>
<td>I/50</td>
</tr>
<tr>
<td>Parameter plotted as abscissa</td>
<td>λωRα</td>
<td>λωγI</td>
<td>λωγI</td>
</tr>
<tr>
<td>Diameter of circles centered on abscissa</td>
<td>λωγI</td>
<td>λωγI</td>
<td>λωγI</td>
</tr>
</tbody>
</table>

**LASER RESONATORS**

The resonators used in laser oscillators usually take the form of an open structure consisting of a pair of spherical mirrors facing each other, as shown in Figure 14-7. The mirror spacing is d, and the radii of curvature of the mirrors are R₁ and R₂.

**Stability**

A resonator is stable if

\[ 0 < \left(1 - \frac{d}{R₁}\right) \left(1 - \frac{d}{R₂}\right) < 1. \]

In a stable resonator, ray bundles traveling back and forth between the mirrors are periodically refocused so that energy remains in the resonator. However, no refocusing occurs in unstable resonators and so a relatively large part of the energy escapes from the resonator on each traversal.

**STABILITY DIAGRAM**. In Figure 14-8, each resonator type is represented by a point. Unstable systems are represented by points in the shaded areas.

**Resonator Modes**

A mode of a laser resonator is defined as a slowly decaying electromagnetic field configuration whose relative distribution does not change with time. It can be represented by a wave propagating back and forth between the mirrors.

**DIFFRACTION LOSSES.** The modal field decays because the resonator is an open structure and light escapes from it. The loss can be thought of as due to diffraction at the finite apertures of the mirrors when the wave propagates back and forth in between.

**FUNDAMENTAL BEAT FREQUENCY.** The frequency spacing \( ν₀ \) between neighboring (longitudinal) resonances of the laser cavity is

\[ ν₀ = c/2d, \]

where c is the light velocity.

**The Integral Equations of Laser Resonators**

Within the approximations that the mirrors are large compared to the wavelength and that the field in the resonator is substantially transverse electromagnetic, the Fresnel-Kirchhoff formulation of Huygen's principle can be used to derive integral equations that relate the fields at the two opposing mirrors of the resonator; the solutions to the equations are the modal fields and their losses. In general, the equations take the form

\[ \gamma^{(1)} E^{(1)}(s₁) = \int S₁ \, K^{(1)}(s₁, s₂) E^{(2)}(s₂) s₂, \]

\[ \gamma^{(2)} E^{(2)}(s₂) = \int S₂ \, K^{(2)}(s₁, s₂) E^{(1)}(s₁) s₁, \]

where the integrations are taken over the mirror surfaces \( S₁ \) and \( S₂ \), respectively. The subscripts and superscripts one and two denote mirrors one and two; \( s₁ \) and \( s₂ \) are symbolic notations for transverse coordinates on the mirror surface, e.g., \( s₁ = (x₁, y₁) \) and \( s₂ = (x₂, y₂) \) or \( s₁ = (r₁, \phi₁) \) and \( s₂ = (r₂, \phi₂) \); \( K^{(1)} \) and \( K^{(2)} \) give the attenuation and phase shift suffered by the wave in transit from one mirror to the other; the kernels \( K^{(1)} \) and \( K^{(2)} \) are functions of the distance between \( s₁ \) and \( s₂ \) and, therefore, depend on the mirror geometry;
they are equal \([K^{(s)}(x)z) = K^{(s)(y)}z)\) but, in general, are not symmetric \([K^{(s)}(x)z) \neq K^{(s)(y)}z)\).

**ORTHOGONALITY OF THE MODES.** The field distribution functions corresponding to the different mode orders are orthogonal over the respective mirror surfaces; that is

\[
\int_{S_1} K^{(m)}(x)\overline{K^{(n)}(x)}dS_1 = 0, \quad m \neq n
\]

and

\[
\int_{S_2} K^{(m)}(x)\overline{K^{(n)}(x)}dS_2 = 0, \quad m \neq n.
\]

where \(m\) and \(n\) denote different mode orders. It is to be noted that the orthogonality relation is non-Hermitian and is the one that is generally applicable to lossy systems.

**MODE NUMBERS.** The modes are distinguished by their mode numbers, which are \(m, n, q\) and \(p, \ell\) for rectangular geometries and \(p, l\) and \(q\) for cylindrical geometries. The mode number \(q\) measures the number of field zeros of the standing-wave pattern along the \(z\) axis. In a rectangular geometry, the transverse-mode numbers \(m\) and \(n\) measure the field nodes in the \(x\) and \(y\) directions. In a circular geometry the transverse mode numbers \(p\) and \(l\) measure the nodes in the \(r\) and \(\phi\) coordinates.

**INTEGRAL EQUATIONS FOR RESONATORS WITH SPHERICAL MIRRORS.** When the mirrors are spherical and have rectangular or circular apertures, the two-dimensional integral equations can be separated and reduced to one-dimensional equations. In the case of rectangular mirrors, the one-dimensional equations in Cartesian coordinates are the same as those for infinite-strip mirrors; for the \(x\) coordinate, they are

\[
y^{(1)}w^{(1)}(x) = \int_{-\infty}^{\infty} K(x, x_2)w^{(1)}(x_2)dx_2
\]

and

\[
y^{(2)}w^{(1)}(x) = \int_{-\infty}^{\infty} K(x, x_2)w^{(2)}(x_2)dx_2,
\]

where the kernel \(K\) is given by

\[
K(x, x_2) = \frac{1}{2\pi} \exp\left(-\frac{ik}{2}(g_1x_1^2 + g_2x_2^2 - 2x_1x_2)\right).
\]

Similar equations can be written for the \(y\) coordinate, so that \(E(x, y) = \text{Re}\{x\}(y)\) and \(y = y_2y_1\). As shown in Figure 14-9, \(a_1\) and \(a_2\) are the half-widths of the mirrors in the \(x\) direction, \(d\) is the mirror spacing, \(k = 2\pi/\lambda\), and \(\lambda\) is the wavelength. The radii of curvature of the mirrors \(R_1\) and \(R_2\) are contained in the factors

\[
g_1 = 1 - \frac{d}{R_1}, \quad g_2 = 1 - \frac{d}{R_2}.
\]

In the case of circular mirrors the equations are reduced to a one-dimensional form by using cylindrical coordinates and by assuming a sinusoidal azimuthal variation of the field; that is, \(E(r, \phi) = R_2(e^{i\phi})\). The radial distribution functions \(R^{(1)}\) and \(R^{(2)}\) satisfy the one-dimensional integral equations:

\[
y^{(1)}R^{(1)}(r_1) = \int_{r_1}^{\infty} K(r_1, r_2)R^{(1)}(r_2)dr_2,
\]

and

\[
y^{(2)}R^{(2)}(r_1) = \int_{r_1}^{\infty} K(r_1, r_2)R^{(2)}(r_2)dr_2,
\]

where the kernel \(K\) is given by

\[
K(r_1, r_2) = \frac{1}{4R_1R_2} \int_{r_1}^{\infty} \int_{r_1}^{\infty} \exp\left(-\frac{ik}{2d}(g_1r_1^2 + g_2r_2^2 - 2r_1r_2)\right) J_0\left(\frac{d}{R_2}g_1r_1^2 + g_2r_2^2\right) dr_1dr_2,
\]

and \(J_0\) is a Bessel function of the first kind and 0th order. Here \(a_1\) and \(a_2\) are the radii of the mirror apertures and \(g_1\) and \(g_2\) are defined as above.

**FRENSÉL NUMBER.** The Fresnel Number \(N\) is defined as

\[
N = a_1a_2/d\lambda.
\]

It is a key parameter in the solution of the above integral equations.

**Approximate Mode Patterns.** Within certain approximations the modes of spherical mirror resonators are TEM waves with the transverse distributions given below. The approximations are valid for stable resonators, and as long as the predicted energy distribution is well confined within the apertures of the mirrors.

**MODE PATTERNS FOR RECTANGULAR GEOMETRY.** Rectangular geometry can be imposed by mirrors of square or rectangular aperture. For this geometry the transverse field distribution \(E(x, y)\) of a TEM \(_{mn}\) mode is approximately described by

\[
E(x, y) = E_0H_n(\sqrt{2}x/a)H_m(\sqrt{2}y/a) \exp\left(-\frac{x^2 + y^2}{w^2}\right),
\]

where \(E_0\) is a constant amplitude factor. \(H_n(x)\) is the Hermite polynomial of \(n\)th order defined by

\[
H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (2x)^k x^{n-k}/\sqrt{n!}.
\]

where \([n/2] = n/2\) or \((n - 1)/2\), according to whether \(n\) is even or odd. Some Hermite polynomials of low order are

\[
H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x.
\]

The parameter \(w\) which appears in these expressions is the beam radius or spot size; formulas for \(w\) will be given below. Observed mode patterns of some of the lower order modes are shown in Figure 14-10a.
MODE PATTERNS FOR CIRCULAR GEOMETRY. In systems with a circular geometry, cylindrical coordinates \((r, \phi, z)\) are used. The transverse field distribution \(E(r, \phi)\) of a TEM\(_{m,n}\) mode is approximately given by

\[
E(r, \phi) = E_0 \left( \frac{r}{W} \right)^p L_p^m (\frac{r^2}{W^2}) \exp \left( - \frac{r^2}{W^2} \right) \left( \sin \frac{\phi}{\phi_0} \right),
\]

where \(W\) is again the beam radius. \(L_p^m(x)\) is the generalized Laguerre polynomial defined by

\[
L_p^m(x) = \frac{x^{p-m}}{p!} \frac{d^p}{dx^p} \left( e^{-x} x^m \right).
\]

Some low-order Laguerre polynomials are

\[
L_0^0(x) = 1 \\
L_1^1(x) = \frac{x + 1}{2} \\
L_2^2(x) = \frac{(x + 1)(x + 2) - (l + 2)x + \frac{1}{2}x^2}{2}.
\]

Modes of circular geometry are not commonly observed in practice unless the resonator is carefully adjusted to be axi-symmetric. Some of the observed low-order modes are shown in Figure 14-10b.
The resonant condition is

\[ \nu \nu_0 = (\nu + 1) + \frac{1}{\pi}(m + n + 1) \arctan \sqrt{(1 - d/R_1)(1 - d/R_2)}, \]

where the square root should be given the sign of \((1 - d/R_2)\), which is equal to the sign of \((1 - d/R_1)\) for a stable resonator.

**Polarization of the Modes**

Field patterns of linearly polarized modes are shown in Figure 14-13. Since the modes are approxim-
Exact Mode Patterns for \( N = 1 \)

The mode patterns obtained from a numerical solution of the resonator integral equation are shown below. They are for the case of symmetrical resonators with circular mirrors and a Fresnel number of \( N = \alpha^2/\rho d = 1 \). The data are shown for various values of the parameter

\[
g_1 = g_2 = g = 1 - d/R.
\]

Figure 14-15 shows the transverse amplitude and phase distributions of the fundamental TEM\(_{00}\) mode. Figure 14-16 shows the amplitude and phase distributions of the TEM\(_{11}\) mode.

**Diffraction Losses and Phase Shifts**

The **diffraction loss** of a resonator mode is defined as the fractional energy lost per transit due to diffraction effects at the mirrors. The **phase shift** \( \beta \) is the phase shift suffered by the mode in transit from one mirror to the other, in addition to the geometrical phase shift, which is given by \( 2\pi d/\lambda \). The resonant frequency of a mode is given by

\[
u R = (q + 1) + \beta/\pi.
\]

The diffraction losses for the two lowest order (TEM\(_{00}\) and TEM\(_{11}\)) modes of a stable resonator with a pair of identical, circular mirrors \( (g_1 = g_2, g_1 = g = g) \) are given in Figures 14-17 and 14-18 as functions of the Fresnel number \( N \) and for various values of \( g \). The curves are obtained by a numerical solution of the resonator integral equations. Corresponding curves for the phase shifts are shown in
Figures 14-19 and 14-20. The horizontal portions of the phase shift curves can be calculated from the formulas

\[ \beta = (2p + 1 + l) \arccos \sqrt{g/g_2} \]
\[ = (2p + 1 + l) \arccos g_1, \quad \text{for } g_1 = g_2. \]

It is to be noted that the loss curves are applicable to both positive and negative values of \( g \), while the phase-shift curves are for positive \( g \) only; the phase shift for negative \( g \) is equal to 180 degrees minus that for positive \( g \).

Equivalent Resonator Systems

**Basic Resonator Parameters.** Resonators of the kind shown in Figure 14-9 are characterized by three basic parameters, \( N, G_1 \) and \( G_2 \), which are defined as

\[ N = a_1a_2/\lambda d \quad \text{(Fresnel number)} \]
\[ G_1 = a_1^2/a_2^2 - (1 - d/R) \]
\[ G_2 = a_1^2/a_2^2 - (1 - d/R_2). \]

If these three parameters are the same for any two resonators, then they have the same diffraction loss, the same resonant frequencies, and mode patterns that are scaled versions of each other. Equivalent resonators can be obtained by changing the curvature of the mirrors and at the same time their apertures.

**G-Factor Reversal.** The diffraction loss and the intensity pattern of a resonator mode remain invariant if both \( g_1 \) and \( g_2 \) are reversed in sign; the eigenfunctions \( E \) and the eigenvalues \( \gamma \) of the integral equations merely take on complex conjugate values. An example of such equivalent systems is that of plane-parallel \( (g_1 = g_2 = 1) \) and concentric \( (g_1 = g_2 = -1) \) resonator systems.

**Resonator with Internal Optical System.** A resonator with a general optical system of large aperture inserted between the mirrors has an equivalent resonator without the optical system in it. The basic parameters of this equivalent resonator are given by

\[ N = a_1^2a_2^2/\lambda B \]
\[ G_1 = a_1^2/a_2^2 \left( A - \frac{B}{R} \right) \]
\[ G_2 = a_1^2/a_2^2 \left( D - \frac{B}{R_2} \right). \]
where $a_l$ and $a_r$ are the radii of the mirror apertures, $R_1$ and $R_2$ are the radii of curvature of the mirrors, and $A, B, C,$ and $D$ are the ray matrix elements of the internal optical system.

As an example, consider a resonator with an internal lens of focal length $f$, as shown in Figure 14-21. If the lens aperture is large enough, this system is equivalent to a lensless resonator with the following basic parameters:

$$N = \frac{a_la_r}{\lambda(d_1 + d_2 - d_1d_2/f)}$$

$$G_1 = \frac{a_1}{a_2} \left( 1 - \frac{d_1}{f} - \frac{1}{R_1} \left( d_1 + d_2 - \frac{d_1d_2}{f} \right) \right)$$

$$G_2 = \frac{a_2}{a_1} \left( 1 - \frac{d_2}{f} - \frac{1}{R_2} \left( d_1 + d_2 - \frac{d_1d_2}{f} \right) \right),$$

where $d_1$ and $d_2$ are the distances between the lens and the mirrors.

A further example is the symmetric resonator with an internal limiting aperture in its central plane, as shown in Figure 14-22(a). The mirrors are assumed to be infinitely large. Such a resonator is equivalent to a Fabry-Perot resonator with a lens in its middle, as shown in Figure 14-22(b). The latter, by virtue of the example given above, has an equivalent as shown in Figure 14-22(c), with the basic parameters given by

$$N = \frac{\sigma^2}{\lambda d(1 - d/2R)}$$

$$G_1 = G_2 = 1 - \frac{d}{R}.$$