11 Synchronization of Chaotic Systems

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11.1 Introduction

The first observation of a synchronization phenomenon in physics is attributed to C. Huygens (1673) during his experiments for developing improved pendulum clocks (Huygens, 1673). Two clocks hanging on the same beam of his room were found to oscillate with exactly the same frequency and opposite phase due to the (weak) coupling in terms of the almost imperceptible oscillations of the beam generated by the clocks. Huygens also observed that both clocks were able to synchronize only in those cases where their individual frequencies almost coincided - a prerequisite that turned out to be typical for synchronization of periodic oscillations.

Since Huygens' early observation synchronization phenomena were discovered and practically used by many physicists and engineers. Rayleigh investigated synchronous oscillations of vibrating organ tubes and electrically or mechanically connected tuning forks in acoustics (Rayleigh, 1945) and Van der Pol and Van der Mark constructed radio tube oscillators where they observed entrainment when driving such oscillators sinusoidally (Van der Pol and Van der Mark, 1926).\textsuperscript{1} These phenomena were investigated theoretically by Van der Pol (Van der Pol, 1927). Synchronization plays also an important role in celestial mechanics where it explains the locking of revolution periods of planets and satellites. Furthermore, synchronization was not only observed in physics but also in (neuro-) biology where rhythms and cycles may entrain (Glass and Mackey, 1988) or synchronizing clusters of firing neurons are considered for being crucial for information processing in the brain (Schechter, 1996; Schiff et al., 1996).

In this and other cases synchronization can play a functional role because it establishes some special relation between coupled systems. For periodic oscillations, for example, the frequencies of interacting systems may become the same or lock with a rational ratio due to synchronization. More complex relations can be expected and have been found for coupled chaotic systems. At first glance, synchronization of chaotic systems seems to be rather surprising because one may naively expect that the sensitive dependence on initial conditions would lead to

\footnote{\textsuperscript{1}This was probably also one of the first experiments where chaotic dynamics was observed, although the authors didn’t investigate this aspect.}
an immediate breakdown of any synchronization of coupled chaotic systems. This, however, is not the case. Before possible meanings of the term "chaos synchronization" will be discussed, an example is given for illustration. This example consists of two bi-directionally coupled Lorenz systems (Lorenz, 1963; Ott, 1993; Schuster, 1995):

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1) - c(x_2 - y_2) \\
\dot{y}_1 &= \sigma(y_2 - y_1) + c(x_2 - y_2) \\
\dot{x}_2 &= r x_1 - x_2 - x_1 x_3 \\
\dot{y}_2 &= r y_1 - y_2 - y_1 y_3 \\
\dot{x}_3 &= x_1 x_2 - b x_3 \\
\dot{y}_3 &= y_1 y_2 - b y_3.
\end{align*}
\tag{11.1.1}
\]

with \( \sigma = 10, r = 28, \) and \( b = 2.666. \) The parameter \( c \) is a coupling constant. Numerically we find that for \( c > 1.96 \) both Lorenz systems (11.1.1) synchronize (\( \lim_{t \to \infty} ||x(t) - y(t)|| = 0 \)) and generate the well-known Lorenz attractor. For \( c = \sigma/2 \) this can even be proved analytically. In this case the difference \( e_1 = y_1 - x_1 \) converges to zero, because \( e_1 = -\sigma e_1. \) For \( x_1 = y_1 \) the synchronization errors \( e_2 = y_2 - x_2 \) and \( e_3 = y_3 - x_3 \) of the remaining variables also vanish, because the Lyapunov function \( L = e_2^2 + e_3^3 \) decreases monotonously with \( \frac{1}{2} \dot{L} = -e_2^2 - be_3^3 < 0. \)

Both coupled dynamical systems of this example were assumed to be exactly the same or identical. Therefore, it is possible to observe identical synchronization (IS) where the states of both Lorenz systems converge to the same trajectory and \( x(t) - y(t) \to 0 \) for \( t \to \infty. \) Of course, in general two coupled systems are not exactly the same or even may be of completely different origin (e.g., an electrical circuit coupled to a mechanical system). What does "synchronization" mean in such a more general case? Different periodic systems are usually called synchronized if either their phases are locked,

\[
|n\phi_1 - m\phi_2| < \text{const}, \quad n, m \in \mathbb{N} \tag{11.1.2}
\]

or the weaker condition of frequency entrainment

\[
\omega_1 : \omega_2 = n : m, \quad n, m \in \mathbb{N} \tag{11.1.3}
\]

holds where \( \omega_1 = \langle d\phi_1/\text{dt}\rangle_t \) and \( \omega_2 = \langle d\phi_2/\text{dt}\rangle_t \) are the natural frequencies of both oscillators. For chaotic systems the notions of "frequency" or "phase" are in general not well defined and can thus not be used for characterizing synchronization.\(^2\) But there is another feature of synchronized periodic oscillations that can also be found with coupled chaotic systems: the dimension of the attractor in the combined state space of the coupled systems reduces as soon as the coupling exceeds some threshold values. For two periodic oscillators synchronization leads to a transition from a quasiperiodic (torus) attractor (incommensurate frequencies, dimension 2) to a common periodic orbit (dimension 1). In systems consisting of \( N \) coupled periodic oscillators with \( N \) independent frequencies a similar transition from an \( N \)-dimensional torus attractor to lower dimensional tori (partial entrainment) and finally to a common periodic orbit may occur if some effective coupling is increased.

\(^2\)For some class of chaotic systems a phase variable can be introduced and used to quantify phase synchronization, see Sec. 11.6.4.
In this sense synchronization of periodic systems means that the dimension of the resulting attractor of the coupled system is smaller than the sum of the dimensions of all individual attractors of the elements. This notion of synchronization can of course also be applied to coupled chaotic systems. Landa and Rosenblum suggest to call two coupled chaotic systems to be synchronized if the dimension of the attractor in the combined state space equals the dimensions of its projections into the subspaces corresponding to the coupled systems (Landa and Rosenblum, 1993).

A reduction of the dimension means that the degrees of freedom are reduced for the coupled system due to the interaction of its components. In particular, for extended systems consisting of many coupled elements (e.g., nonlinear oscillators) the investigation of synchronization phenomena may thus result in a better understanding of the typical occurrence of low-dimensional cooperative dynamics.

Dimension reduction due to coupling is an important indicator for (chaos) synchronization but turned out to be a too coarse-grained tool to be able to resolve important details of the relation between the coupled systems. Due to the coupling there may exist, for example, a function mapping states of one of the coupled systems to those of the other. This feature is called generalized synchronization (see Sec. 11.6) and its occurrence cannot be proved using dimension estimates of the attractors involved. Also the occurrence of certain (transversal) instabilities leading to intermittent breakdown of synchronization (see Sec. 11.3) is difficult to characterize with any dimension analysis. It is one goal of this article to give an overview of typical synchronization phenomena of chaotic systems and concepts for describing and understanding them.

Both systems of the previous example are bi-directionally coupled. This is, however, not necessary, because chaos synchronization occurs already for uni-directionally coupled systems. This case is in some sense easier to investigate and may be viewed as an important building block for understanding bi-directionally coupled systems. Furthermore, uni-directionally coupled systems are important for potential applications of chaos synchronization in communication systems (transmitter-receiver or encoder-decoder pairs) or time series analysis where the information flow is also in a single direction only. Therefore, only uni-directionally coupled (chaotic) systems are investigated in the following. In some cases it is evident how the presented concepts and results can be extended to bi-directionally coupled systems and in others not. Although the case of uni-directional coupling is already very rich of interesting phenomena we expect that bi-directional coupling leads to even more complex dynamics.

In the presentation of chaos synchronization of uni-directionally coupled systems we start in Sec. 11.2 with pairs of identical systems. Although in physical reality two systems will in general never happen to be exactly the same, this case is very useful to explain basic features of chaos synchronization and to show how synchronizing pairs can be derived from a given chaotic model. Furthermore, identical systems play an important role with some of the potential applications of chaos synchronization that will be discussed in Sec. 11.7. From a more general
point of view synchronization phenomena between different dynamical systems are discussed in Sec. 11.6 in terms of generalized synchronization and phase synchronization. Generalized synchronization can lead to the existence of a function that maps (asymptotically for $t \to \infty$) states of the drive system to states of the response system. In this case the chaotic dynamics of the response system can be predicted from the drive system. A weaker notion of synchronization is phase synchronization where only a (partial) synchronization of some phase variable occurs while amplitudes may remain uncorrelated. In Sec. 11.5 we demonstrate that chaos synchronization can also be achieved for spatially extended systems like coupled nonlinear oscillators or partial differential equations. Here, a typical goal is synchronization with a minimum of information flow from the drive to the response system and we make use of the concept of sporadic driving that is introduced in Sec. 11.4.

In Sec. 11.7 we briefly discuss potential applications of chaos synchronization in communication systems and time series analysis. In chaos-based encoding schemes the synchronization is used to recover a signal that is necessary to decode the message and in the field of data analysis model parameters are estimated by minimizing the synchronization error.

The problem of controlling (chaotic) dynamical systems can also be investigated in the framework of synchronization. In this case one of the coupled systems is typically implemented on an analog or digital computer (called controller) that is connected to an experimental or technical device. The main goal of control is to make this device to follow some prescribed dynamics given by the controller. In this sense the controlled process synchronizes with the controller and this may happen for bi-directional as well as uni-directional coupling (open-loop control). If the control algorithm is based on the knowledge of the (full) state of the dynamics a (nonlinear) state observer is necessary to recover the information about the current state of the process from some (scalar) time series. Such a state observer may be considered as a dynamical system that is implemented on a computer and that synchronizes with some external process due to a uni-directional coupling. Note that in contrast to synchronization phenomena between (experimental) physical systems here we a have a "free choice" how to implement the dynamical system which is implemented on the controller with full access to all state variables and parameters. Therefore, rather powerful synchronization methods can be implemented that allow to recover the state of the external process provided a sufficiently exact model exists that may be based on first principles or empirical approximations.

**11.2 Synchronization of identical systems**

In this chapter the synchronization features of pairs of identical systems are investigated. Although in physical systems and experiments coupled systems are in general never exactly the same, this idealized case is of importance for developing suitable concepts for describing chaos synchronization. These concepts and results can directly be applied to systems which are (almost) identical including configu-
rations that are of practical interest as will be discussed in more detail in Sec. 11.7. They also play an important role in the case of generalized synchronization of non-identical systems that will be treated in Sec. 11.6.

Pioneering work on chaos synchronization was done by Fujisaka and Yamada (Fujisaka and Yamada, 1983), Pikovsky (Pikovsky, 1984), Afraimovich, Verichev and Rabinovich (Afraimovich et al., 1986) and Pecora and Carroll (Pecora and Carroll, 1990) who presented the first examples of synchronization of uni-directionally coupled chaotic systems. This work had a strong impact and stimulated very intense research activities on chaos synchronization and related questions (He and Vaidya, 1992; Murali and Lakshmanan, 1994; Parlitz et al., 1996). Identical synchronization of chaotic systems was also demonstrated experimentally in particular using electronic circuits (Anishchenko et al., 1992; Rulkov et al., 1992; Kocarev et al., 1992) and laser systems (Roy et al., 1994; Sugawara et al, 1994; Tsukamoto et al., 1996). Synchronization properties of chaotic phase-locked loops have been investigated in Refs. (Endo, 1991; De Sousa Vieira et al., 1991, 1992, 1994a and 1994b).

11.2.1 Constructing pairs of synchronizing systems

For some practical applications and for studying chaos synchronization it is useful to derive pairs of synchronizing systems from a given chaotic model.

The most popular method for constructing synchronizing (sub-) systems was introduced by Pecora and Carroll (Pecora and Carroll, 1990; Carroll and Pecora, 1993). With that approach a given dynamical system

\[ \dot{u} = g(u) \]

is decomposed into two subsystems

\[ \dot{v} = g_v(v, w) \]
\[ \dot{w} = g_w(v, w). \]

with \( v = (u_1, ..., u_k) \) and \( w = (u_{k+1}, ..., u_N) \) such that any second system

\[ \dot{w'} = g_w(v, w') \]

that is given by the same vector field \( g_w \), the same driving \( v \), but different variables \( w' \) synchronizes (\( ||w' - w|| \to 0 \)) with the original \( w \)-subsystem. The coupling is uni-directional and the \( v \)-system and the \( w \)-system are referred to as the drive system and the response system, respectively. Note that only a finite number of possible decompositions exists that is bounded by the number of different subsystems \( N(N - 1)/2 \). In general, only a few of the possible response subsystems possess negative conditional Lyapunov exponents and may be used to implement synchronizing systems using the method of Pecora and Carroll.

This method and almost all other coupling schemes for achieving synchronization can formally be described as a decomposition of a given (chaotic) system into
an active and a passive part, where different copies of the passive part synchronize when driven by the same active component (Kocarev and Parlitz, 1995; Parlitz et al., 1996). Consider an arbitrary $N$-dimensional (chaotic) dynamical system

$$\dot{z} = F(z).$$

The goal is to rewrite this autonomous system as a non-autonomous system that possesses certain synchronization properties. Formally, we may write

$$\dot{x} = f(x, s)$$

where $x$ is the new state vector corresponding to $z$ and $s$ is some vector valued function of time given by

$$s = h(x) \quad \text{or} \quad \dot{s} = h(x, s).$$

The pair of functions $f$ and $h$ constitutes a decomposition of the original vector field $F$ (see also the example that follows). The crucial point of this decomposition is that for suitable choices of the function $h$ any system

$$\dot{y} = f(y, s)$$

that is given by the same nonautonomous vector field $f$, the same driving $s$, but different variables $y$, synchronizes with the original system (11.2.2), i.e., $\|x - y\| \to 0$ for $t \to \infty$. More precisely, synchronization of the pair of (identical) systems (11.2.2) and (11.2.4) occurs if the dynamical system describing the evolution of the difference $e = y - x$,

$$\dot{e} = f(y, s) - f(x, s) = f(x + e, s) - f(x, s)$$

possesses a stable fixed point at the origin $e = 0$. In some cases this can be proved using stability analysis of the linearized system for small $e$,

$$\dot{e} = Df_z(x, s) \cdot e,$$

or using (global) Lyapunov functions. In general, however, the stability has to be checked numerically by computing so-called transversal or conditional Lyapunov exponents (CLEs) using the linearized equation (11.2.5). Synchronization occurs if all conditional Lyapunov exponents of the nonautonomous system (11.2.2) are negative. In this case system (11.2.2) is a passive system and we call the decomposition an active-passive decomposition (APD) of the original dynamical system (11.2.1).

3If $S$ is given by a (static) function $S = h(x)$ then $X = Z$, but for coupling signals $S$ that are generated by an ODE $S = h(s, x)$ the dimension of $X$ may be smaller than that of $Z$.

4The notion transversal is due to the fact that $e = y - x$ describes the motion transversal to the synchronization manifold $x = y$. Using (11.2.5) one obtains the Lyapunov exponents of the response system under the condition that it is driven by the signal $s$ (Pecora and Carroll, 1990). For uni-directionally coupled systems the CLEs are a subset of the Lyapunov spectrum of the coupled system (see Appendix A).

5In the presence of noise this condition is not sufficient as will be discussed in Sec. 11.3.
The stability of the passive parts does not exclude chaotic solutions. To illustrate the APD we consider a decomposition of the Lorenz model that may be written as:

\[
\begin{align*}
\dot{x}_1 &= -10x_1 + s(t) \\
\dot{x}_2 &= 28x_1 - x_2 - x_1x_3 \\
\dot{x}_3 &= x_1x_2 - 2.666x_3
\end{align*}
\]

with

\[s(t) = 10x_2.\]

The corresponding response system is given by:

\[
\begin{align*}
\dot{y}_1 &= -10y_1 + s(t) \\
\dot{y}_2 &= 28y_1 - y_2 - y_1y_3 \\
\dot{y}_3 &= y_1y_2 - 2.666y_3.
\end{align*}
\]

To estimate the temporal evolution of the difference vector \(e = y - x\) of the states of the two systems we note that the difference \(e_1 = y_1 - x_1\) of the first components converges to zero, because \(e_1 = -10e_1\). Therefore, the remaining two-dimensional system describing the evolution of the differences \(e_2 = y_2 - x_2\) and \(e_3 = y_3 - x_3\) can, in the limit \(t \to \infty\), be written as:

\[
\begin{align*}
\dot{e}_2 &= -e_2 - x_1(t)e_3 \\
\dot{e}_3 &= x_1(t)e_2 - 2.666e_3.
\end{align*}
\]

Using the Lyapunov function \(L = e_2^2 + e_3^2\) it can be shown that \(\dot{L} = -2(e_2^2 + 2.666e_3^2) < 0\). This means that the synchronization is globally stable provided that other perturbations like noise added to the driving signal or parameter mismatch between drive and response can be excluded (or are at least of very small magnitude) (Brown et al., 1994). The CLEs of this decomposition are given by \(\lambda_1 = -1.805, \lambda_2 = -1.861\) and \(\lambda_3 = -10\) with respect to the natural logarithm.

As it was outlined in Ref. (Parlitz et al., 1996) APD-based synchronization methods are also very closely related to the open-loop control method proposed by Hübler and Lüscher (Hübler and Lüscher, 1989; Jackson and Hübler, 1990) and Pyragas’ chaos control approach (Pyragas, 1993; Kittel et al., 1994). In general, synchronization and controlling chaos are very closely related although the goals may sometimes be different (Kapitaniak, 1994). In particular for the synchronization of identical systems, many methods that have been developed in control theory for nonchaotic dynamics may also be applied to chaotic systems. Instead of decomposing a given chaotic system one may also synthesize it starting from a stable linear system \(\dot{x} = A \cdot x\) where some appropriate nonlinear function \(s = h(x)\) is added such that the complete system

\[
\dot{x} = A \cdot x + s
\]

is chaotic (Wu and Chua, 1993). It is easy to verify that in this case the error dynamics is given by the stable system \(\dot{e} = A \cdot e\), and synchronization occurs for all
initial conditions and arbitrary signals $s$. In this way synchronized chaotic systems may be designed with specific features for applications.

Another approach to construct synchronizing systems is based on cascades of low-dimensional systems. In this way one may generate hyperchaotic systems that can be synchronized using a single scalar signal (Kocarev and Parlitz, 1995; Parlitz et al., 1996; Güémez and Matías, 1995 and 1996).

Synchronizing hyperchaotic systems have also been presented in Refs. (Peng et al., 1996; Tamaševićius and Čenys, 1997) and occur typically when spatially extended dynamics are investigated (see Sec. 11.2).

The function $h$ that defines the coupling signal may depend not only on the current state $x(t)$ of the drive system but also on its pre-history. In this way linear filters may be incorporated in the coupling scheme, such that the coupling signal possesses some specific spectral properties (e.g. bandlimited for a bandlimited transmission channel).

In view of potential applications where *amplitude quantization* of the driving signal occurs (e.g. due to an A/D-converter) the case of a step function $h$ has also been investigated (Stojanovski et al., 1996a and 1997b). It turned out that synchronization is always possible, but in general the chaotic dynamics turned into periodic oscillations when the number of discretization levels becomes (too) small.

In the previous discussion of synchronization methods, the function $s$ was assumed to be vector valued in general. For the examples and in the following, however, we will consider only cases with scalar signals $s$ that are most interesting for practical applications of synchronization.

Usually when studying synchronization phenomena all parameters of the systems involved are kept fixed. In the case of parameter mismatch, however, the synchronization error typically increases quite rapidly with the parameter differences. In such a case one might ask: "Can I find an additional dynamical system for the parameters of the response system such that they adapt automatically to the parameter values of the drive in order to achieve (perfect) synchronization?" The answer is "yes", and this case is called *autosynchronization* or *adaptive synchronization* (Mossayebi et al., 1991; John and Amritkar, 1994; Caponetto et al., 1995; Dedieu and Ogorzalek, 1995; Parlitz, 1996; Parlitz et al., 1996; Chua et al., 1996, Cazelles et al., 1996).

11.3 Transversal instabilities and noise

In Sec. 11.2 it was stated that synchronization occurs, if all conditional Lyapunov exponents (CLEs) of the response system are negative. This condition turned out not to be sufficient for many cases in the following sense (Ashwin et al., 1994; Gauthier and Bienfang, 1996; Venkataramani et al., 1996; Yang, 1996; Lai et al., 1996;
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Heagy et al., 1996). Negative CLEs with respect to the chaotic dynamics of the drive mean that the synchronization manifold \( M = \{(x, y) : x = y\} \) is attracting on average. There may, however, exist locations on \( M \) where it is repelling, and when the orbit comes close to such a region, it is pushed away from the synchronization manifold and synchronization breaks down. As long as the orbit of the coupled system lies exactly on the synchronization manifold, such a local transversal instability at some point \((x_0, y_0)\) has no effect. Practically, however, typical orbits are never on but only close to \( M \), because they converge asymptotically from some initial condition to \( M \) and/or noise kicks them off the synchronization manifold (Brown et al., 1994a). In these cases any trajectory moves away from \( M \) as long as it stays in the region of transversal instability. The effective growth of the synchronization error \( e = x - y \) depends on the size of this region, the strength of the transversal instability, and the period of time the orbit spends in the unstable region. As soon as the trajectory leaves the region of instability it is attracted again by the synchronization manifold and thus an averaged negative CLE exists that implies \( e \to 0 \) in the noiseless case. With (arbitrary small) amounts of noise (added to the coupling signal) the difference \( ||e|| \) cannot (on average) become smaller than some constant \( \delta \) which is given by the noise level.

In this way the action of the contracting part of the synchronization manifold is limited, and short or long passages through the repelling regions will lead to small or large amplifications of the minimum distance \( \delta \) given by the noise. Since sufficiently long trajectory segments in the repelling region occur only unfrequently the breakdown of synchronization due to transversal instabilities is an intermittent phenomenon. This can be seen in the example given in Fig. 11.1 that was computed using a uni-directionally coupled pair of chaotic one-dimensional Gaussian maps:

\[
\begin{align*}
x^{n+1} &= f(x^n) \\
y^{n+1} &= f(y^n) + c[f(x^n) + r^n - f(y^n)]
\end{align*}
\]

where \( f(x) = \exp(-a^2 [x - b]^2) \) with \( a = 3.5, b = 0.5, c = 0.4 \). The largest CLE of this system is negative for \( c > 0.3 \). In order to stimulate the intermittent bursts of the synchronization error \( e^n = y^n - x^n \), uniformly distributed noise \( r^n \in [-10^{-5}, 10^{-5}] \) is added to the driving signal. The origin of the regions of transversal instability are for example unstable fixed points or unstable periodic orbits (UPOs) of the driving system\(^7\) that fail to entrain the corresponding fixed points or periodic orbits (POs) of the response system. When driven with one of these UPOs the counterparts of the response system possess positive CLEs and no identical synchronization of the UPOs occurs. Such unstable response fixed points or orbits may occur, for instance, due to period doubling bifurcations.

To illustrate this mechanism we consider the fixed point \((x_F, x_F) = (0.6781, 0.6781)\) of the coupled system (11.3.1) that is located on the synchronization manifold \( M \). The stability features of this fixed point are given by the

\(^7\)Any typical chaotic attractor contains an infinite number of unstable periodic orbits (Ott, 1993; Schuster, 1995) that may also be used for driving and synchronizing a response system (Gupte and Amritkar, 1993).
eigenvalues \( \mu_1 = f'(x_F) \) and \( \mu_2 = (1 - c)f'(x_F) \) of the Jacobian matrix of (11.3.1) at \((x_F, x_F)\). The first eigenvalue \( \mu_1 \) describes the instability within the synchronization manifold and does not depend on the coupling. The second eigenvalue, however, reflects the transversal (in)stability and depends on \( c \). For \( c > 0.662 \) the stability criterion \(|\mu_2| < 1\) holds, and at the critical value \( c_{PD} \approx 0.662 \) a period doubling bifurcation occurs. Thus for \( c < c_{PD} \) the unstable fixed point (period-1 UPO) of the drive fails to entrain a fixed point of the response but leads to a period-2 cycle (at least for values of \( c \) that are sufficiently close to \( c_{PD} \). If \( c \) is decreased furthermore, a complete period doubling cascade occurs.) For \( c = 0.4 \), for example, the response of the fixed point \( x_F \) is given by the period-2 orbit \(..., 0.778, 0.505,...\) with \( \text{CLE} \lambda = -0.32 \). Such a subharmonic entrainment of periodic orbits turns out to be also of importance for generalized synchronization as will be discussed in Sec. 11.6. Note that subharmonic entrainment means that some stable periodic response occurs in contrast to a possible chaotic response to the periodic UPO-driving.

Another way to illustrate the transversal instability is presented in Fig. 11.2 where the gray shaded regions consist of points \((x, y)\) that are moved away from the synchronization manifold by the map (11.3.1). The thick dot on the diagonal gives the position of the fixed point \((x_F, x_F)\) and the other two thin dots represent a period-2 UPO of the driving map. For a coupling with \( c = 0.4 \) (that was also used for Fig. 11.1) two large regions of transversal instability occur (Fig. 11.2a) that contain the period-1 and the period-2 UPO. If the coupling constant is increased to \( c = 0.666 \) the unstable region shrinks to two small spots as can be seen in Figure 11.2b. For this value of the coupling all UPOs entrain periodic orbits on the synchronization manifold with negative CLEs. The remaining small gray shaded areas indicate regions where positive local Lyapunov exponents occur. For \( c \geq 0.667 \) the gray shaded regions vanish and the synchronization manifold is everywhere transversally stable.
Finally, we would like to note that typically the phenomenon of **riddled basins** is associated with transversal instabilities if another attractor exists off the synchronization manifold. In this case attractor basins occur which are of positive measure but contain no open sets, i.e. in any neighbourhood of a point of the basin of the first attractor are points of the basin of the second attractor (see Refs. (Alexander et al., 1992; Ott et al., 1993; Parmenter and Yu, 1994; Ashwin et al., 1994; Heagy et al., 1994; Ott and Sommerer, 1994; Lai et al., 1996; Ding and Yang, 1996; Venkataramani et al., 1996; Lai and Grebogi, 1996) for details).

### 11.4 Sporadic driving

To achieve synchronization of two *continuous* systems it is not necessary to couple them continuously. Even if the coupling is switched on at *discrete* times $t_n = nT$ only, synchronization may occur if the coupling and the time interval $T$ are suitably chosen (Amritkar and Gupte, 1993; Stojanovski et al., 1996a, 1997a and 1997b; Chen, 1996; Parlitz et al., 1997). This kind of **sporadic driving** leads to synchronization, for example in those cases where for the given coupling signal the corresponding *continuous* driving would lead to synchronization. In this case synchronization occurs for all $T < T_S$ where $T_S$ is a threshold value that depends on the dynamical system and the particular coupling. In general, sporadic driving may be defined in the following way. Let

\[
\dot{x} = g(x) \tag{11.4.1}
\]

and

\[
\dot{y} = g(y) \tag{11.4.2}
\]

be two continuous dynamical systems that are given by the same vector field $g$ but different state vectors $x$ and $y$, respectively. Like in the previous sections system
(11.4.1) will be called drive and system (11.4.2) response. Let the signal be a scalar function of the state of the drive that is, however, available only at discrete times \( t_n = nT \) where \( T \) is some sampling time. In order to describe how the resulting discretely sampled time series \( \{s^n\} \) with \( s^n = s(t_n) = h(x(t_n)) \) is used to drive the response system (11.4.2) we consider now both continuous systems as discrete systems that are given by the flow \( \phi^T \):

\[
\begin{align*}
x^{n+1} &= \phi^T(x^n) \\
y^{n+1} &= \phi^T(y^n).
\end{align*}
\]

The flow \( \phi^T \) is obtained by integrating the ODEs (11.4.1) and (11.4.2) over the period of time \( T \) where \( x^n = x(t_n) \) and \( y^n = y(t_n) \) are the states of the drive and the response system at time \( t_n \), respectively. During this period of time both systems are not coupled and run freely. To achieve (chaos) synchronization of drive and response we apply now the concept of an active-passive decomposition (APD) to the discrete systems (11.4.3) and (11.4.4) and rewrite the map \( x^{n+1} = \phi^T(x^n) \) as

\[
x^{n+1} = f(x^n, s^n) \tag{11.4.5}
\]

where \( s^n = h(x^n) \). An example for such a formal decomposition is:

\[
f(x^n, s^n) = \phi^T(x^n + [s^n - h(x^n)]c) \tag{11.4.6}
\]

where \( c \) is a vector containing some coupling constants. If the APD is successful the new (formally) nonautonomous system (11.4.5) is passive, i.e., it possesses only negative conditional Lyapunov exponents when driven by \( \{s^n\} \). This, however, implies that any copy of (11.4.5)

\[
y^{n+1} = f(y^n, s^n) \tag{11.4.7}
\]

will synchronize \( \|x^n - y^n\| \to 0 \) for \( n \to \infty \). To illustrate this type of coupling we will use in the following as drive and response two Lorenz systems that are given by:

\[
\begin{align*}
\dot{x}_1 &= 10(x_2 - x_1) \\
\dot{x}_2 &= 28x_1 - x_2 - x_1x_3 \\
\dot{x}_3 &= x_1x_2 - 2.666x_3.
\end{align*} \tag{11.4.8}
\]

The coupling signal \( s^n \) is \( x_2(t_n) \) and the APD is given as

\[
x^{n+1} = f(x^n, s^n) = \phi^T \begin{pmatrix} x_1^n \\ s^n \\ x_3^n \end{pmatrix} \tag{11.4.9}
\]

where \( x^n = x(t_n) \) and \( \phi^T \) denotes the flow generated by the Lorenz system. Both Lorenz systems run independently during a period of time \( T = 0.4 \). Then the variable \( y_2 \) of the response system is replaced by \( x_2 \) of the drive, i.e., the discrete coupling takes place. After this coupling the systems oscillate independently again
11.4 Sporadic driving

Figure 11.3 Synchronization of two chaotic Lorenz systems (11.4.8) due to sporadic driving. Shown is the response variable $y_2$ as a function of time $t$. The vertical dashed lines denote the times $t_n = nT = n \cdot 0.4$ when the coupling is active.

and so on. Figure 11.3 shows the variable $y_2$ of the response system. The times $t_n$ where the coupling takes place are denoted by the vertical dashed lines and the diagram shows the oscillation after some synchronization transient. The time evolution of $y_2(t)$ coincides exactly with the corresponding evolution of the drive variable $x_2(t)$ (not shown here). The synchronization due to the sporadic driving does not only lead to a convergence $|x_2(t_n) - y_2(t_n)| \to 0$ at the coupling times $t_n$ for $n \to \infty$, but also to a perfect interpolation of the time evolution of all state variables between the coupling times. With this coupling the largest conditional Lyapunov exponent is negative for coupling times $T \in [0,0.45]$. For other systems more than one $T$-interval with negative CLEs have been observed (Stojanovski et al., 1997a) and in general it can be shown that for sporadic driving with a $k$-dimensional driving signal (here: $k = 1$) the $k$ smallest CLEs of the response system equal $-\infty$ (Stojanovski et al., 1997a).

Numerical simulations have shown that it becomes more difficult (or even impossible) to find a suitable discrete APD of the flow if $T$ is chosen too large. On the other hand, sporadic driving may lead to synchronization in cases where the corresponding continuous coupling fails (Amritkar and Gupte, 1993; Stojanovski et al., 1996a and 1997a).

Another important issue is the sensitivity of synchronization with respect to noise added to the driving signal. Very similar to the case of continuous coupling synchronization due to sporadic driving may lead to either high quality synchronization or intermittency phenomena depending on the dynamical systems and details of the coupling (Stojanovski et al., 1997a). Furthermore, synchronization due to sporadic coupling can be realized using bandlimited channels for the coupling signal despite the fact that chaotic spectra typically possess no finite support (Stojanovski et al., 1997a). The fact that in this way chaotic signals can be transmitted through bandlimited channels is not in contradiction with the well-known sampling theorem since the chaotic signals are generated by deterministic differential equations which in addition are known at the receiver. Only the initial
conditions of the driving system are not known at the receiving end of the channel, i.e., at the interpolating side. The sampling theorem, on the other hand, assumes no knowledge about the information source.

Another feature of sporadic driving which is important for practical implementations is the fact that synchronization can also be achieved when the coupling is switched on for short but finite periods of time (Stojanovski et al., 1997a). Therefore, the basic mechanism seems to be robust enough to be observed also in real physical systems where instantaneous variations of some variable are impossible.

In the following sections two potential applications of sporadic driving will be discussed: synchronization of spatially extended systems (Sec. 11.5) and parameter identification from time series (Sec. 11.7).

11.5 Spatially extended systems

Spatially extended systems like coupled oscillators or partial differential equations are usually governed by (very) high dimensional chaotic attractors. Nevertheless, it is possible to achieve synchronization of such hyperchaotic systems if they are coupled in a suitable way\(^8\) (see Refs. (Lai and Grebogi, 1994; Gang and Zhilin, 1994; Kocarev et al., 1997)). From a practical point of view, however, some constraints apply because often a coupling at all spatial locations and at all times is not feasible. Instead of such a continuous coupling in space and time it is desirable to use discrete coupling schemes where the coupling is active only at certain locations and/or at discrete times. Discrete coupling not only simplifies experimental realizations of synchronized spatio-temporal dynamics but also leads to a reduction of the information flow from the driving system to the response system that may be used to characterize the underlying chaotic dynamics or to store the full chaotic evolution in terms of the control signals.

To illustrate the synchronization of spatially extended systems we shall consider now two coupled Kuramoto-Sivashinsky (KS) equations (Hyman and Nicolaenko, 1986)

\[
\begin{align*}
    u_t + 4u_{xxxx} + \gamma \left( u_{xx} + \frac{1}{2} (u_x)^2 \right) + \frac{dw}{dt} &= 0, \\
    \frac{dw}{dt} &= -\frac{\gamma}{2\pi} \int_0^{2\pi} (u_x)^2 dx.
\end{align*}
\]

Here \( x \in [0,2\pi] \) with periodic boundary conditions and Eq. (11.5.2) is used to subtract the meanvalue from the dynamics. Figure 11.4a shows the spatio-temporal chaos that occurs for \( \gamma = 200 \). In order to couple Eq. (11.5.1) to another KS-equation we assume that we can measure 10 signals \( s_k^t \) from \( k = 1, \ldots, 10 \) sensors that

\(^8\)Here we are considering uni-directionally coupled pairs of spatially extended systems. Internal synchronization phenomena within such systems have also been observed but are beyond the scope of this article.
are equidistantly distributed in the $x$-interval $[0, 2\pi]$. These sensors are assumed to possess a finite resolution such that they measure a local average of the dynamics. The sensor signal at position $x^k = k \cdot 2\pi/10$ and time $t_n = nT$ is given by

$$s^k_u(t_n) = \int_{x^k - \Delta x}^{x^k + \Delta x} u(x, t_n) \, dx$$

where the width $\Delta x = 4\pi/25$ specifies the spatial resolution of the sensor and $T$ is the period of time between the coupling events. The same set of sensor signals $\{s^k_v(t_n)\}$ is assumed to exist for the response KS-equation that is identical to the drive and generates a solution $v(x, t)$. The discrete coupling consists in an updating of the $v$-values in the $k = 1, ..., 10$ sensor intervals $[x^k - \Delta x, x^k + \Delta x]$ at times $t_n = nT$ with some coupling constant $c$:

$$\forall n = 1, 2, 3, ... \forall k = 1, ..., 10$$
$$\forall x \in [x^k - \Delta x, x^k + \Delta x]:$$
$$\tilde{v}(x, t_n) = v(x, t_n) + c \left[ s^k_v(t_n) - s^k_u(t_n) \right]$$

where $\tilde{v}$ stands for the value of $v$ immediately after the moment when the coupling was active. Figure 11.4b shows the temporal evolution of the response KS-equation and in Fig. 11.4c we have plotted the synchronization error $|v(x, t) - u(x, t)|$. The spatially averaged synchronization error as a function of time is given in Figure 11.4d.
11.6 Synchronization of nonidentical systems

Until now we have considered pairs of identical systems where *identical synchronization* \( \lim ||x(t) - y(t)|| = 0 \) may occur. If two different systems are coupled identical synchronization is in general not possible (i.e. not a solution of the coupled system), but other types of synchronization may be observed. In this section we present and compare different definitions of *generalized synchronization* (GS) that have been proposed during the last years.\(^9\)\(^10\)

### 11.6.1 Generalized synchronization I

For chaotic systems Afraimovich et al. (Afraimovich et al., 1986) gave the first definition for what was later called *generalized synchronization* (GS) by Rulkov et al. (Rulkov et al., 1995). In their definition Afraimovich et al. called two systems synchronized if in the limit \( t \to \infty \) a homeomorphic function exists mapping states of one system to states of the other (including some time shift \( \alpha(t) \) with \( \lim (t + \alpha(t))/t = 1 \)). Later the assumption of a homeomorphism was relaxed and two uni-directionally systems are said to be in synchrony if their states \( x \) and \( y \) are asymptotically related by some function \( H \) so that \( ||H(x(t)) - y(t)|| \to 0 \) for \( t \to \infty \). This definition of *generalized synchronization* (GS) was used in Refs. (Kocarev and Parlitz, 1996; Hunt et al., 1997) and can be verified using time series based methods as suggested by Rulkov et al. (Rulkov et al., 1995). Mathematically it may be formulated as follows:

**Def. I:**

*Generalized synchronization* of the uni-directionally coupled systems

\[
\begin{align*}
\text{drive} \quad \dot{x} &= f(x) \quad (x \in \mathbb{R}^n) \\
\text{response} \quad \dot{y} &= g(y,x) \quad (y \in \mathbb{R}^m)
\end{align*}
\]  

occurs for the attractor \( A_x \subset \mathbb{R} \) of the drive system if an attracting *synchronization set* \( M = \{(x, y) \in A_x \times \mathbb{R}^m : y = H(x)\} \) exists that is given by some function \( H : A_x \to A_y \subset \mathbb{R}^m \) and that possesses an open *basin* \( B \supset M \) such that:

\[
\lim_{t \to \infty} ||y(t) - H(x(t))|| = 0 \quad \forall (x(0), y(0)) \in B.
\]

An analogous definition can be given for discrete dynamical systems. Whether the function \( H \) is continuous or even smooth depends on the features of the drive and response system and the attraction properties of the set \( M \) (Davies, 1996;

\(^9\)Closely related are investigations of IIR-filters, see (Davies and Campbell, 1996; Stark, 1997) and the references cited therein.

\(^{10}\)In this section we consider only ideal systems without noise. In the presence of noise similar effects like those discussed in Sec. 11.3 have to be taken into account.
11.6 Synchronization of nonidentical systems

Hunt et al., 1997; Stark, 1997).

This definition may be motivated by the requirement that statements about synchronization should be independent of the coordinate system used. As an example we consider two uni-directionally coupled systems:

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= rx_1 - x_2 - x_1x_3 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{y}_1 &= \sigma(x_2 - f^{-1}(y_1))/(f^{-1})'(y_1) \\
\dot{y}_2 &= f^{-1}(y_1)(ra - y_3) - y_2 \\
\dot{y}_3 &= f^{-1}(y_1)y_2 - by_3
\end{align*}
\]

where \(f^{-1}\) and \((f^{-1})'\) denote the inverse and its derivative, respectively, of an invertible function \(f : \mathbb{R} \to \mathbb{R}\). 11 To prove that for this pair of systems GS occurs with \(H(x) = (f(x_1), ax_2, ax_3)\) we consider first the difference \(e_1 = f^{-1}(y_1) - x_1\).

Using Eq. (11.6.1) it is easy to show that \(\dot{e}_1 = -\sigma e_1\) and thus \(e_1 \to 0\) for \(t \to \infty\) if \(\sigma > 0\). Therefore, asymptotically \(f^{-1}(y_1) = x_1\) or \(y_1 = f(x_1)\). For the remaining two-dimensional system given by \(e_2 = y_2 - ax_2\) and \(e_3 = y_3 - ax_3\) one can show with a Lyapunov function \(L = (e_2^2 + e_3^2)/2\) that \(\dot{L} = -e_2^2 - be_3^2 < 0\) for \(b > 0\) and thus \(y - H(x) \to 0\) for all initial conditions. In this case the synchronization set \(M\) is a globally attracting submanifold with \(y = H(x)\) and its basin of attraction \(B\) is the whole product space \(\mathbb{R}^n \times \mathbb{R}^m\) of both state spaces. This example is based on example (11.2.6)-(11.2.7) and was constructed starting from a pair of two identical Lorenz systems that synchronize. Then, the response system was subject to a change of the coordinate system given by \(H\). Since any diffeomorphic transformation doesn’t change stability properties the new synchronization manifold (here: \(y = H(x)\)) remains stable. Thus identical synchronization implies GS in any diffeomorphic equivalent coordinate system. On the other hand, if GS is observed between two dynamical systems with a diffeomorphic function \(H\) this function can be used to perform a change of the response coordinate system such that in the new coordinate system the response system synchronizes identically with the drive system.

To check for which values of the coupling parameter \(\alpha\) GS occurred one may apply nearest neighbors statistics (Rulkov et al., 1995; Pecora et al., 1995) to detect the existence of a continuous function relating states of the drive to states of the response. This approach for identifying generalized synchronization can be applied to uni- and bi-directionally coupled systems if the original (physical) state spaces of drive and response are accessible. If only (scalar) time series from the drive and the response system can be sampled, then delay embedding (Takens, 1980; Sauer, 1991) may be used to investigate neighborhood relations in the corresponding reconstructed state spaces (Rulkov et al., 1995). In this case, however, only generalized synchronization of uni-directionally coupled systems can be detected by predicting the (reconstructed) state of the response system using a time series from the drive system! In the opposite direction a prediction of the evolution of the drive system based on data from the response system is always possible (i.e.

\footnote{As an example the reader may substitute \(f(x_1) = \exp(x_1)\) with \(f^{-1}(y_1) = \ln(y_1)\) and \((f^{-1})'(y_1) = 1/y_1\).}
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with and without generalized synchronization), because (almost) any time series measured at the response system may also be viewed as a time series from the combined systems \textit{drive and response} and may thus be used to reconstruct and predict the dynamics of drive and response.\footnote{Note that \textit{without} GS a time series of the drive system contains no information about the response system.}

Generalized synchronization in the sense of Def. I can also be found or established using the following result (Kocarev and Parlitz, 1996):

\textbf{Proposition:}

GS occurs for an attractor $A$ of the coupled systems (11.6.1) if an open basin of synchronization $B \subset \mathbb{R}^n \times \mathbb{R}^m$ exists with $M \subset B$ such that for all initial values $(x_0, y_0) \in B$ the driven system $\dot{y} = g(y, x)$ is \textit{asymptotically stable} in the sense that

$$\forall (x_0, y_{10}), (x_0, y_{20}) \in B : \lim_{t \to \infty} \|y(t, x_0, y_{10}) - y(t, x_0, y_{20})\| = 0.$$ 

Some remarks concerning this proposition are in order:

(i) The construction of $H$ in the proof given in Ref. (Kocarev and Parlitz, 1996) is based on the uniqueness of the inverse of the flow generated by (11.6.1). Therefore, the analogous proposition for discrete systems holds only if the drive is given by an \textit{invertible} map.

(ii) The assumptions of the proposition are not fulfilled for \textit{subharmonic entrainment} of periodic orbits, because in this case different basins of attraction occur. An entrainment with ratio $T_D : T_R = 1 : p$ ($p > 1$), for example, results in $p$ basins for the initial values of the response system. For the proposition, however, it is assumed that a single basin $B$ exists. This exclusion of subharmonically entrained periodic orbits is necessary, because for these solutions $H$ is \textit{not} a function. If, for example, a periodic orbit of the drive entrains a stable periodic orbit of the response with twice the period (i.e. $T_D : T_R = 1 : 2$) then any point on the attractor of the drive is mapped to \textit{two} points on the response orbit and $H$ is in this case a relation but \textit{not} a function. This multivaluedness always occurs for \textit{subharmonic periodic entrainment} with $T_D < T_R$. Recall that subharmonic entrainment was also the origin of transversal instabilities as discussed in Sec. 11.3.

(iii) Asymptotical stability (as defined in the proposition) can be proved analytically using Lyapunov functions. Numerically this condition can be checked by computing the (largest) Lyapunov exponent of the response system. In this case, however, it has to be made sure \textit{additionally} that all (unstable) periodic orbits of the drive entrain a stable periodic orbit of the response system \textit{with the same period}, because (U)POs that lead to subharmonic entrainment or even chaotic response orbits cannot be excluded using a stability
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analysis based on (globally averaged) Lyapunov exponents. Practically, this requirement can be checked using nearest neighbors statistics.

(iv) In Ref. (Pyragas, 1996) it was argued that $H$ is a smooth function if the conditional Lyapunov exponents of the response system are smaller than the smallest Lyapunov exponent of the drive, because in this case the Lyapunov dimension of the attractor of the coupled system would not depend on the driven response system. This condition seems, however, not to be sufficient, because (i) for some unstable periodic orbits it may locally be not fulfilled resulting in a local loss of smoothness (Hunt et al., 1997) and (ii) there may also exist subharmonically entrained orbits where $H$ is not a function (Parlitz et al., 1997).

In order to illustrate the proposition given above we show with this example that GS occurs for a uni-directionally coupled system consisting of a Rössler system driving a Lorenz system. The equations of the drive and the response system are:

\begin{align*}
\dot{x}_1 &= 2 + x_1(x_2 - 4) & \dot{y}_1 &= -\sigma(y_1 - y_2) \\
\dot{x}_2 &= -x_1 - x_3 & \dot{y}_2 &= ru(t) - y_2 - u(t)y_3 \\
\dot{x}_3 &= x_2 + 0.45x_3 & \dot{y}_3 &= u(t)y_2 - by_3
\end{align*}

where $u(t)$ is an arbitrary scalar function of $x_1$, $x_2$, $x_3$, and $\sigma, b > 0$. In order to show that GS occurs we consider the difference $e = y - y'$, where the primed variables describe a second trajectory of the response system starting from different initial conditions. Using the Lyapunov function $L = (e_1^2/a + e_2^2 + e_3^2)/2$ one obtains $\dot{L} = -e_1e_2 + e_1^2 - e_2^2 - be_3^2 = -(e_1 - e_2/2)^2 - 3e_2^2/4 - be_3^2 < 0$, i.e. the response system is asymptotically stable for arbitrary drive signals $u$ and arbitrary initial conditions. Therefore, GS always occurs although drive and response are completely different systems.

11.6.2 Generalized synchronization II

There are many cases where drive and response are not related by a function but a weaker notion of synchronization applies that may be defined as follows\footnote{This definition is motivated by the auxiliary system method of Abarbanel et al. (Abarbanel et al., 1996).}:

\textbf{Def. II: Generalized synchronization} of uni-directionally coupled systems

\textbf{Drive} \quad \dot{x} = f(x) \quad (x \in \mathbb{R}^n)

\textbf{Response} \quad \dot{y} = g(y, x) \quad (y \in \mathbb{R}^m)

occurs if there exists an open synchronization basin $B \subset \mathbb{R}^n \times \mathbb{R}^m$ such that

\[ \forall (x_0, y_{10}), (x_0, y_{10}) \in B : \lim_{t \to \infty} ||y(t; x_0, y_{10}) - y(t; x_0, y_{20})|| = 0. \]

This definition says that GS occurs if the response system is asymptotically stable with respect to the driving signal and at first glance the definition may look very
similar to the proposition given in the previous section. But there is a crucial difference, because we do not assume here that the complete attractor is contained in the basin $B$. Therefore, this definition for synchronization includes also the case of subharmonic entrainment of periodic oscillations where several basins coexist (for entrainment with $T_D : T_R = 1 : p$, $p$ basins $B_i \ (i = 1, ..., p)$ occur). Practically, the occurrence of this type of synchronization can be checked by computing the CLEs of the response system. Again, a rigorous investigation should include not only the (averaged) CLEs of the chaotic dynamics but also the Lyapunov exponents of (all) response orbits that are generated by UPOs embedded in the chaotic drive attractor. If there exists an UPO of the drive that leads to a chaotic response, for example, then two response trajectories will diverge when the drive comes close to this UPO.

Another practical method for checking the existence of GS (in the sense of Def. II) that is a direct implementation of the criterion of asymptotic stability was suggested by Abarbanel et al. (Abarbanel et al, 1996). It is based on the use of a second auxiliary response system. The investigated pair of coupled systems is said to have the property of GS if starting from different initial conditions both response systems converge to the same trajectory. The advantage of this approach is the fact that it can directly be applied to experimental systems provided a second copy of the response system is available like for electronic circuits (Rulkov, 1996; Rulkov and Sushchick, 1996) or laser systems. If both response systems differ slightly one even obtains additional informations about the robustness of the synchronization.

### 11.6.3 Non-identical synchronization of identical systems

Usually (generalized) synchronization is studied if two different systems $f \neq g$ are coupled but it may also occur with identical systems $f = g$ that show no identical synchronization ($x \neq y$). As an example, we consider two uni-directionally coupled identical Lorenz systems

$$
\begin{align*}
\dot{x}_1 &= 10(x_2 - x_1) \\
\dot{x}_2 &= 28x_1 - x_2 - x_1x_3 \\
\dot{x}_3 &= x_1x_2 - 2.666x_3 \\
\dot{y}_1 &= 10(y_2 - y_1) + c(x_1 - y_1) \\
\dot{y}_2 &= 28y_1 - y_2 - y_1y_3 \\
\dot{y}_3 &= y_1y_2 - 2.666y_3.
\end{align*}
$$

For $c > 7.7$ this coupled system shows the wellknown identical synchronization. There exists, however, also a parameter interval for the coupling $c$ where no identical but generalized synchronization in the sense of Def. II occurs. Figure 11.5 shows an example for this case. Figs. 11.5a and Fig. 11.5b show the variables $x_1$ and $y_1$ of the drive and the response system, respectively, and in Fig. 11.5d the synchronization error $|x_1(t) - y_1(t)|$ is plotted. It is obvious that there is no identical synchronization. Figure 11.5c shows the dynamics of a second (auxiliary) response system that started with different initial conditions. After some transient both response systems (Figs. 11.5b and 11.5c) synchronized mutually as can also be seen in Fig. 11.5e. Usually when using conditional Lyapunov exponents for verifying
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For synchronization of identical systems the Jacobian matrix of the vector field is computed using the state vector of the drive system. These Lyapunov exponents will be called IS-LEs in the following. However, for investigating the mutual synchronization of a pair of response systems the Jacobian matrix has to be determined using a response state. The resulting set of Lyapunov exponents are related to GS and are therefore called GS-LEs. If identical synchronization occurs both sets of LEs coincide but not for generalized synchronization. This difference is illustrated in Fig. 11.6a where the largest IS-LE is plotted as a dotted curve and the maximum GS-LE is given by the solid line. For coupling values $c < -6.7$ the GS-LE becomes negative and GS occurs while the IS-LE remains positive. The onset of GS can also be seen when computing the averaged drive-response and response-response synchronization errors $E$, respectively that are given in Fig. 11.6b.
Figure 11.6 Uni-directionally coupled Lorenz systems. (a) GS-LE (solid) and IS-LE (dotted) conditional Lyapunov exponents detecting response and identical synchronization, respectively. (b) Averaged synchronization errors of drive and response (dotted) and between both response systems (solid).

11.6.4 Phase synchronization

Another generalization of the notion of identical synchronization is the phenomenon of phase synchronization (PS) (Stone, 1992; Rosenblum et al., 1996; Parlitz et al., 1996; Pikovsky et al., 1996, 1997; Osipov et al., 1997; Fabiny et al., 1993). It is best observed when a well defined phase variable can be identified in both coupled systems. This can be done heuristically for strange attractors that spiral around some particular point (or "hole") in a two-dimensional projection of the attractor. In such a case, a phase angle $\phi(t)$ can be defined that decreases monotonically. Phase synchronization of two coupled systems occurs if the difference $|\phi_1(t) - \phi_2(t)|$ between the corresponding phases is bounded by some constant. This phenomenon may be used in technical or experimental applications where a coherent superposition of several output channels is desired (Fabiny et al., 1993; Roy and Thornburg, 1994).

In more abstract terms PS occurs when a zero Lyapunov exponent of the response system becomes negative. This leads to a reduction of the degree of freedom of the response system in the direction of the flow. For systems where a phase variable can be defined the direction of the flow coincides in general with the coordinate that is described by the phase variable. A zero LE that becomes negative reflects in this sense a restriction that is imposed on the motion of the phase variable. If the zero LE that decreases is the largest LE of the response system then phase synchronization occurs together with GS. If there exist, however, in addition to the formerly zero LE, other LEs which are and remain positive, PS occurs but no GS.

Another phenomenon that is closely related to PS is lag synchronization that was observed only recently by Rosenblum et al. (Rosenblum et al., 1997) and leads to synchronization with some time delay between drive and response. For more details see the contribution of Pikovsky et al. in this volume.

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$^{14}$A more general definition includes rational relations $|n\phi_1 - m\phi_2| < \text{const}$ for arbitrary integers $n$ and $m$. Compare also the definitions (11.1.2) and (11.1.3) for periodic oscillations.
11.7 Applications and Conclusion

During the last six years chaos synchronization has become one of the most intensely studied topics in nonlinear dynamics. This development was mainly stimulated by the seminal paper of Pecora and Carroll (Pecora and Carroll, 1990), although some very interesting results were already found in the 80s (see Sec. 11.2). One reason for the success of the approach suggested by Pecora and Carroll was probably the fact that they mentioned already in their first paper in 1990 the possibility of using uni-directionally coupled systems in communication systems based on chaos. Once a unidirectionally coupled pair of synchronizing systems (drive and response) has been found it can be used in different ways for encoding and masking messages. The basic idea is to transmit an information signal with a broadband chaotic carrier signal and to use synchronization to recover the information at the receiver. Different implementations of this general concept have been suggested (Kennedy, 1997):

(a) Chaotic masking
The information is added to a chaotic carrier and the synchronization of the response system in the receiver is used to recover the message (Kocarev et al., 1992; Cuomo and Oppenheim, 1993; Cuomo et al., 1993; Murali and Lakshmanan, 1993; Lozi and Chua, 1993). An improved scheme based on dissipative pseudorandom dynamics was suggested in (Gershenfeld and Grinstein, 1995).

(b) Chaos modulation
The information signal is contained in the transmitted signal and (in contrast to (a)) drives the transmitter system and the receiver system in exactly the same way (Halle et al., 1993; Volkovskii and Rulkov, 1993; Wu and Chua, 1993; Wu and Chua 1994; Kocarev and Parlitz, 1995; Parlitz et al., 1996). This method is related to the inverse systems approach (Feldmann et al., 1996) and in principle it allows to recover the information exactly.

(c) Chaos shift keying
Binary information signals are encoded by switching between different drive systems. At the receiver the message can be recovered by monitoring the synchronization of the corresponding response systems of the receiver (Parlitz et al., 1992; Dedieu et al., 1993).

(d) Parameter modulation
The information signal is used to modulate a parameter of the drive system and the receiver uses auto-synchronization to recover the messages by reproducing this modulation (Carroll and Pecora, 1993; Parlitz et al., 1996; Parlitz and Kocarev, 1996). Practically this scheme can be implemented by adding a feedback loop to the response system that controls the dynamics of the modulated parameter. The variations of the information signal have to be slow compared to the convergence properties of the parameter controlling loop.
(e) **Encoding using generalized synchronization**

Transmitter and receiver possess a common keysequence that they use to drive (identical) response systems. Due to generalized synchronization the output of the response systems is a (complicated) function of the key sequence that is used to encode and decode the message (Xiao et al., 1996; Parlitz et al., 1997).

An important feature of the masking and modulation methods (a) and (b) is the fact that the frequency spectrum of the chaotic carrier can (and should) be the same as that of the information signal. Both signals can therefore not be separated using linear data analysis tools like linear filters. Using numerical and experimental examples it was demonstrated that synchronization can in principle be used for decoding suitably encoded messages, where “chaos” has the important task to scramble the data such that they cannot be easily deciphered by third parties. On the other hand, for low-dimensional chaos methods have been suggested to break such an encryption (Short, 1994; Pérez and Cerdeira, 1995; Stojanovski et al., 1996b). Whether chaos synchronization based encryption methods will be developed in a way that they can compete with the very powerful algorithms already known in (standard) cryptography will turn out in the future. Important prerequisites for achieving this ambitious goal have been demonstrated to be fulfilled, like synchronizing very highdimensional chaotic systems using only a scalar (discretely sampled) signal.

Another application of synchronization consists in model verification (Brown et al., 1994b) and parameter estimations from time series (Parlitz et al., 1996). Assume that a (chaotic) experimental time series has been measured and that the structure of a model is known, but not the parameters and those state variables that have not been measured. The goal is to find these unknown parameters and perhaps also the time evolution of the variables that have not been measured. This problem can be solved by minimizing the synchronization error (Parlitz et al., 1996) or using auto-synchronization (see Sec. 11.2). In the same way response systems can be established that monitor slow changes of technical devices provided that a sufficiently exact model of the process of interest is available.

In the previous sections different approaches were presented for synchronizing a pair of identical dynamical systems. Although these methods are already quite general they are not succesful in all cases. Given a coupling signal and two systems to be synchronized the Pecora-Carroll approach may, for example, fail, because no appropriate stable subsystem can be found. On the other hand, other coupling schemes may be succesful for the same configuration. One may therefore ask the general question: “For which dynamical systems and for which coupling signals is chaos synchronization possible?” At first glance, it may look hopeless to answer this question. However, one can easily show that in general any pair of uni-directionally coupled dynamical systems can be synchronized using (almost) any (smooth) coupling signal (Stojanovski, 1997c). This is an immediate consequence of the state space reconstruction theorems in nonlinear time series analysis (Takens, 1980; Sauer, 1991). There, delay or derivative coordinates are used to
reconstruct the states of a dynamical process (here: drive system) from scalar time series. More precisely, the reconstructed states are diffeomorphic images of the original states and the theorems provide rather general conditions for the existence of the underlying diffeomorphism. Applying the inverse of this diffeomorphism to the reconstructed states one may thus recover the original states of the drive. The knowledge of these states can then be exploited in different ways to synchronize the response system with the drive, the simplest method being a replacement of the states of the response by the states of the drive at discrete times (Stojanovski, 1997c). The main technical difficulty of such an approach is the computation of the inverse reconstruction map. If a sufficiently large number of pairs \{reconstructed state, original state\} is available one may, for instance, fit an approximating function based on polynomials, radial basis functions or neural networks in order to describe the map: reconstructed state \(\rightarrow\) original state. Using the knowledge about the dynamical equations of the drive system it is also possible to formulate for each state a fixed point problem in the reconstructed state space that may be solved by a (quasi) Newton algorithm. With this method as well as with the approximation method the states of the drive can be computed directly, i.e. without any (exponential) transient like in the case of (identical) synchronization.

The problem of estimating states of a system from measured data has also a long history in control theory where the recovered states are used as input of a controller that tries to drive the system towards some goal dynamics. In this context algorithms that yield the states of the system are called observer and were introduced in 1966 by Luenberger (Luenberger, 1966). The first work on this topic for linear systems dates back to 1960 when Kalman laid the foundations for this field (Ogata, 1990).

Later different methods for (special classes) of nonlinear systems have been proposed (Thau, 1973; Kou et al., 1975; Citarella et al., 1993; So et al., 1994; Morgüll and Solak, 1996) and in the following we give an example for such an approach to synchronization. However, not only observer algorithms from control theory can be used in the context of synchronization. Controlling strategies themselves may be used to drive a response system into a synchronized state (Lai and Grebogi, 1993; Newell et al., 1994 and 1995). The relation between synchronization and (standard) control theory was pointed out by Konnur (Konnur, 1996) (see also (Levine, 1996; Nijmeijer and van der Schaft, 1990; Isidori, 1989)).

In this article we focussed on synchronization methods and mechanisms of uni-directionally coupled systems. Of course, many physical, biological or technical systems consist of bi-directionally interacting elements or components. We hope that the future investigation of different couplings in combination with sophisticated types of synchronization\(^{15}\) will provide a deeper insight into the variety of cooperative phenomena observed in nature.

\(^{15}\)That are yet to be discovered ?!
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Appendix

In the following we show that for a uni-directionally coupled system

\[\begin{align*}
\dot{x} &= f(x) \\
\dot{y} &= g(y, x)
\end{align*}\]  

(11.7.1) (11.7.2)

the conditional Lyapunov exponents of the response system (11.7.2) are a subset of the complete Lyapunov spectrum of the combined system

\[\dot{z} = F(z)\]  

(11.7.3)

with \(z = (y, x)\), \(F = (g, f)\) and \(y \in \mathbb{R}^m, x \in \mathbb{R}^n\). The Lyapunov exponents of (11.7.3) can be computed via a \(QR\)-decomposition of the linearized flow matrix \(Y\) which is a solution of the matrix variational equation

\[\dot{Y} = DF(z(t)) \cdot Y\]  

(11.7.4)

for the initial condition \(Y = I = \text{diag}(1, \ldots, 1)\) (unit matrix). The block structure of the Jacobian matrix \(DF\) of \(F\)

\[DF(z) = \begin{pmatrix} Dg_y & Dg_z \\ 0 & Df_x \end{pmatrix}\]  

(11.7.5)

leads to the same structure for the matrix \(Y\):

\[Y = \begin{pmatrix} Y_A & Y_B \\ 0 & Y_D \end{pmatrix}\]  

(11.7.6)

The fact that the inverse \(R^{-1}\) of the upper triangular matrix \(R\) of the \(QR\)-decomposition is again an upper triangular matrix can now be used to show that the orthonormal matrix \(Q\) possesses also a block structure

\[Q = Y \cdot R^{-1} = \begin{pmatrix} Y_A & Y_B \\ 0 & Y_D \end{pmatrix} \cdot \begin{pmatrix} R_A^{-1} & R_B^{-1} \\ 0 & R_D^{-1} \end{pmatrix}\]  

(11.7.7)

\[= \begin{pmatrix} Y_A \cdot R_A^{-1} & Y_A \cdot R_B^{-1} + Y_B \cdot R_D^{-1} \\ 0 & Y_D \cdot R_D^{-1} \end{pmatrix} = \begin{pmatrix} Q_A & Q_B \\ 0 & Q_D \end{pmatrix}\]  

(11.7.8)
Furthermore, using the identities $Q \cdot Q^{tr} = I = Q^{tr} \cdot Q$ one can show that $Q_B = 0$, and $Q_A$ and $Q_D$ are orthonormal. This gives the matrix equations

$$\begin{align*}
Y_A & = Q_A \cdot R_A \quad (11.7.9) \\
Y_D & = Q_D \cdot R_D. \quad (11.7.10)
\end{align*}$$

Using these equations and the block structure of $DF$ and $Y$ is easy to show that $Y_A$ and $Y_D$ are given by the following matrix variational equations

$$\begin{align*}
\dot{Y}_A & = Dg_y \cdot Y_A \quad (11.7.11) \\
\dot{Y}_D & = Df_x \cdot Y_D. \quad (11.7.12)
\end{align*}$$

These are, however, exactly the ODEs that are used for computing the Lyapunov exponents of the drive (11.7.11) and the conditional Lyapunov exponents of the response system (11.7.12). Since the $QR$-decomposition is unique the diagonal elements of $R_A$ and $R_D$ give the exponents of the drive systems

$$\lambda_i^D = \lim_{t \to \infty} \frac{1}{t} \ln(R_{A,ii})$$

and the conditional Lyapunov exponents of the response system

$$\lambda_j^R = \lim_{t \to \infty} \frac{1}{t} \ln(R_{D,ij})$$

which all together constitute the Lyapunov spectrum of the coupled system (11.7.3).

References

11.7 Applications and Conclusion


122. van der Pol, B. (1927), Phil. Mag. 7-3, 65.


