Nuclei and Particles

An Introduction to Nuclear and Subnuclear Physics

SECOND EDITION
Completely revised, reset, enlarged

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We assume that the potential $U$ decreases faster than $1/r$ and is central. The scattering center is located at the origin of the coordinates. The Schrödinger equation is

$$\nabla^2 \psi + \frac{2m}{\hbar^2} (E - U) \psi = 0$$  \hspace{1cm} (A-1)

We shall try to find a solution of Eq. (A-1) with an asymptotic behavior for $\psi$ given by

$$\psi(r) = e^{ikz} + f(\theta) \frac{e^{ikr}}{r}$$  \hspace{1cm} (A-2)

$$k = p/\hbar \quad \text{or} \quad \nu = k\hbar/m$$  \hspace{1cm} (A-3)

Physically Eq. (A-2) represents a plane wave $e^{ikz}$ of amplitude 1 and a spherical outgoing wave of amplitude $f(\theta)$. 

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The outgoing flux through a large sphere centered at the origin is given by

$$\varphi = \frac{\hbar k}{m} \int |f(\theta)|^2 d\omega$$  \hspace{1cm} (A-4)

as is seen from the expression for current in quantum mechanics. By definition the differential cross section is then

$$|f(\theta)|^2 = \frac{d\sigma}{d\omega}$$  \hspace{1cm} (A-5)

We shall now calculate this quantity. We first need an identity

$$e^{ikz} = \frac{\pi \sqrt{2}}{(kr)^{1/2}} \sum_{l=0}^{\infty} i^l (2l + 1)^{1/2} Y_{l,0}(\theta) J_{l+1/2}(kr)$$  \hspace{1cm} (A-6)

with the spherical harmonic

$$Y_{l,0}(\theta) = \frac{(2l + 1)^{1/2}}{(4\pi)^{1/2}} P_l(\cos \theta)$$  \hspace{1cm} (A-7)

and $P_l(\cos \theta)$ a Legendre polynomial. The $Y_{l,0}(\theta)$ have the orthonormality property

$$\int Y_{l,0}(\theta) Y_{l',0}(\theta) \, d\omega = \delta_{l,l'}$$  \hspace{1cm} (A-8)

The $J_{l+1/2}(x)$ are Bessel functions with the asymptotic expressions

$$J_{l+1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{x^{l+1/2}}{(2l + 1)!!} \quad x \ll 1$$  \hspace{1cm} (A-9)

$$J_{l+1/2}(x) = \left[ \left( \frac{2}{\pi x} \right)^{1/2} \sin \left( x - \frac{\pi l}{2} \right) \right] \quad x \gg 1$$  \hspace{1cm} (A-10)

The identity can be proved by developing $e^{ikz}$ in a series of Legendre polynomials by the usual Fourier method. Figure A-1 shows the first few $j_l(x) = (\pi/2x)^{1/2} J_{l+1/2}(x)$.

We now also develop $f(\theta)$ in spherical harmonics,

$$f(\theta) = \sum_{l} a_l P_l(\cos \theta) = (4\pi)^{1/2} \sum_{l} \frac{a_l}{(2l + 1)^{1/2}} Y_{l,0}(\theta)$$  \hspace{1cm} (A-11)
The functions \( (\pi/2x)^{1/2}J_{l+1/2}(x) = j_l(x) \) for \( l = 0, 1, 2 \). They can also be expressed as
\[
  j_0(x) = \frac{\sin x}{x}; \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}; \quad j_2(x) = \left( \frac{3}{x^3} - \frac{1}{x} \right) \sin x - \frac{3 \cos x}{x^2}
\]
and insert this expression into Eq. (A-2), obtaining asymptotically
\[
  \psi(r, \theta) = \frac{(4\pi)^{1/2}}{r} \sum_l \frac{Y_{l,0}}{(2l+1)^{1/2}} \left[ e^{ikr} \left( a_l - \frac{i}{2} \frac{2l+1}{k} \right) + e^{-ikr} \left( -1 \right)^l \frac{i}{2} \frac{2l+1}{k} \right]
\]
(A-12)

The coefficient of \( e^{ikr} \) represents the amplitudes of outgoing waves of different angular momenta \( l \), and similarly the coefficient of \( e^{-ikr} \) represents the amplitude of ingoing waves. In a stationary state they must be equal in modulus for each \( l \) separately in order to ensure conservation of matter; however, they may have different arguments. We thus have
\[
  -a_l + \frac{i}{2} \frac{2l+1}{k} = e^{2i\theta} \left( \frac{i}{2} \frac{2l+1}{k} \right).
\]
(A-13)

or
\[
  a_l = \frac{2l+1}{2ik} \left( e^{2i\theta} - 1 \right) \equiv (2l+1)A_l.
\]
(A-14)
The real quantity $\delta_l$ is called the phase shift of the $l$th wave and $A_l$ is called the scattering amplitude.

The $\delta_l$ are not determined by the conservation theorem, but by the asymptotic behavior of $\psi(r)$. The radial part $R(r)$ of $\psi(r)$ can be written $u(r)/r$. Insertion in Eq. (A-1) shows that the asymptotic behavior of $u(r)$ for $r \to \infty$ is

$$u_l(r) = \sin \left(kr - (\pi l/2) + \delta_l \right)$$

(A-15)

if, as assumed, $U(r)$ vanishes rapidly enough. On the other hand, for $r$ small, we have

$$u_l''(r) - \frac{l(l+1)}{r^2} u_l + \frac{2m}{\hbar^2} \left[E - U(r)\right] u_l = 0$$

(A-16)

Near the origin $u_l(r) \sim r^{l+1}/(2l+1)!$, and if we can integrate Eq. (A-16), we find, by joining the solutions for $r$ small and $r$ large, the values of $\delta_l$.

Once the $\delta_l$ are known, we find from Eqs. (A-14), (A-11), and (A-5),

$$\frac{d\sigma}{d\omega} = \frac{1}{4k^2} \left| \sum_l (2l+1) P_l(\cos \theta) (e^{2i\delta_l} - 1) \right|^2$$

(A-17)

This can be integrated over the solid angle, and recalling Eq. (A-8), we have

$$\sigma = 4\pi k^2 \sum_l (2l+1) \sin^2 \delta_l$$

(A-18)

We also have the important relation

$$\text{Im} f(0) = \sigma/4\pi \lambda$$

(A-19)

which is obtained from Eqs. (A-11), (A-14), and (A-18), remembering that $P_l(1) = 1$. Equation (A-19) is sometimes called the optical theorem.

At low energy $\delta_0$ alone is important; the scattering is spherically symmetric, with a cross section

$$\sigma = 4\pi k^2 \sin^2 \delta_0 = 4\pi a^2$$

(A-20)

where $a$, called the scattering length, is susceptible to a simple and important geometric interpretation (see Sec. 10-2).

When the potential has a range $r_0$ the phase shifts different from 0 are only those for which $\hbar l < pr_0$. Semiclassically, consider the impact parameter $b$. For a collision to occur, $b$ must be smaller than $r_0$; otherwise the particle passes outside the potential well. Now the angular momentum with respect to
the center of scattering is \( \hbar l = h p \), and this gives

$$\frac{\hbar l}{p} < r_0 \tag{A-21}$$

as a necessary condition for a collision. If

$$\frac{\hbar}{p} = \lambda \gg r_0 \tag{A-22}$$

only waves with \( l = 0 \) will be scattered.

The phase shifts \( \delta_l \) are all important for the description of the collision. With modern computers the numerical integration of Eq. (A-16) pushed to \( r \) values where \( u_l(r) \) becomes a sine function gives \( \delta_l \) directly. We may, however, want an analytical approximation to \( \delta_l \). To obtain it, consider, along with \( u_l(r) \), a function \( v_l(r) \) obeying Eq. (A-16) when \( U(r) = 0 \), with the proper boundary conditions. The function \( v_l(r) \) thus satisfies the equation

$$v_l''(r) - \frac{l(l + 1)}{r^2} v_l + \frac{2mE}{\hbar^2} v_l = 0 \tag{A-23}$$

and vanishes at the origin. The function \( v_l \) explicitly is

$$v_l(r) = k r j_{l-1/2}(kr) = (\pi kr/2)^{1/2} J_{l+1/2}(kr) \tag{A-24}$$

Multiply Eq. (A-16) by \( v_l(r) \), Eq. (A-23) by \( u_l(r) \), subtract, and integrate; these operations yield

$$\int_0^r (u_l'' v_l - v_l'' u_l) \, dr = u_l' v_l - v_l' u_l \bigg|_0^r = \frac{2m}{\hbar^2} \int_0^r U(r) u_l(r) v_l(r) \, dr \tag{A-25}$$

At the lower limit \( u_l(0) = v_l(0) = 0 \). At large \( r \) from the asymptotic expressions of \( u_l \) and \( v_l \), Eqs. (A-15), and Eq. (A-24) one has

$$- k \sin \delta_l = \frac{2m}{\hbar^2} \int_0^\infty U(r) u_l(r) v_l(r) \, dr \tag{A-26}$$

This integral cannot be calculated exactly unless one knows \( u_l(r) \) and thus has already solved the problem, but if the phase shifts are small, we may replace \( u_l(r) \) by \( v_l(r) \) in the spirit of perturbation theory, obtaining

$$\sin \delta_l = - \frac{2mk}{\hbar^2} \int_0^\infty U(r) [j_{l+1/2}(kr)]^2 r^2 \, dr \tag{A-27}$$

If all the phase shifts are small, we can replace \( \sin \delta_l \) by \( \delta_l \) and \( \exp(2i\delta_l - 1) \).
by \(2i\delta_i\), and we may write, from Eqs. (A-11) and (A-14),

\[
f(\theta) = \sum_i \frac{(2l + 1)}{k} \frac{1}{2i} \left[ \exp(2i\delta_i) - 1 \right] P_l(\cos \theta) \approx \sum_i \frac{(2l + 1)}{k} \delta_i P_l(\cos \theta)
\]

Replacing \(\delta_i\) by its expressions, Eq. (A-27), and interchanging sum and integral, we have

\[
-f(\theta) = \sum_i \frac{(2l + 1)}{k} \frac{2mk}{\hbar^2} \int_0^\infty U(r) j_i(kr) j_i(kr)' \, dr \, P_l(\cos \theta)
\]

\[
= \frac{2m}{\hbar^2} \int_0^\infty U(r) r^2 \, dr \, \sum_l (2l + 1) P_l(\cos \theta) j_i(kr)^2 = (\sin Kr)/Kr \quad \text{(A-28)}
\]

An identity proved in Whittaker and Watson and other mathematical treatises says that

\[
\sum_l (2l + 1) P_l(\cos \theta) j_i(kr)^2 = (\sin Kr)/Kr \quad \text{(A-29)}
\]

where \(K = 2k \sin(\theta/2)\). Replacing this expression in Eq. (A-28), we obtain

\[
-f(\theta) = \frac{2m}{\hbar^2 K} \int_0^\infty U(r) r \sin(Kr) \, dr \quad \text{(A-30)}
\]

the famous Born formula.

There are more direct ways to prove Born's formula. This one shows its relation to phase shifts, as well as the condition of its validity. If the potential is limited to magnitudes of the order of \(U_0\) and extends on a radius \(a\), the condition of validity may be written as \(U_0a/\hbar \nu \ll 1\), where \(\nu\) is the velocity of the particle.

Born's formula (A-30) may be rewritten in a form that is also valid relativistically:

\[
\frac{d\sigma}{d\omega} = |f(\theta)|^2 = \frac{1}{4\pi^2 \hbar^4} \frac{p^2}{v^2} \left| \int_0^\infty U(r) \exp[i(k - k') \cdot r] \, dr \right|^2 \quad \text{(A-31)}
\]

where \(\hbar k, \hbar k'\) are the momenta before and after collision.