OPTICS
More on Geometrical Optics

The preceding chapter, for the most part, dealt with paraxial theory as applied to thin spherical lens systems. The two predominant approximations were, rather obviously, that we had thin lenses and that first-order theory was sufficient for their analysis. Neither of these assumptions can be maintained throughout the design of a precision optical system but both, taken together, provide the basis for a first rough solution. This chapter will carry things a bit further by examining thick lenses and aberrations; even at that, it is only a beginning. The advent of computerized lens design requires a certain shift in emphasis—there is little need to do what a computer can do better. Moreover, the sheer wealth of existing material developed over centuries demands a bit of judicious pruning to avoid a plethora of pedantry.

6.1 THICK LENSES AND LENS SYSTEMS

Figure 6.1 depicts a thick lens, i.e., one whose thickness is by no means negligible. As we shall see, it could equally well be envisioned more generally as an optical system allowing of the possibility that it consists of a number of simple lenses, not merely one. The first and second focal points, or if you like, the object and image foci, \( F_o \) and \( F_i \), can conveniently be measured from the two (outermost) vertices. In that case we have the familiar front and back focal lengths denoted by f.f.l. and b.f.l. When extended, the incident and emerged rays will meet at points, the locus of which forms a curved surface that may or may not reside within the lens. The surface, approximating a plane in the paraxial region, is termed the principal plane [see Section 6.3.1(ii)]. Points where the primary and secondary principal planes (as shown in Fig. 6.1) intersect the optical axis are known as the first and second principal points, \( H_1 \) and \( H_2 \) respectively. They provide a set of very useful references from which to measure several of the system parameters. We saw earlier (Fig. 5.19, p. 110) that a ray traversing the lens through its optical center emerges parallel to the incident direction. Extending both the incoming and outgoing rays until they cross the optical axis locates what are called the nodal points, \( N_1 \) and \( N_2 \) in Fig. 6.2. When the lens is surrounded on both sides by the same medium, generally air, the nodal and principal points will be coincident. The six points, two focal, two principal and two nodal, constitute the cardinal points of the system. As shown in Fig. 6.3 the principal planes can indeed lie completely outside of the lens system. Here although differently configured, each lens in either group has the same power. Observe that in the symmetrical lens the principal planes are, quite reasonably, symmetrically located. In the case of either the planar-concave or planar-convex
lens, one principal plane is tangent to the curved surface—as should be expected from the definition (applied to the paraxial region). By way of contrast, the principal points may certainly be external for meniscus lenses. One often speaks of this succession of shapes having the same power as exemplifying lens bending. A rule of thumb for ordinary glass lenses in air is that the separation $H_1H_2$ roughly equals one third the lens thickness $V_1V_3$.

The thick lens can be treated as consisting of two spherical refracting surfaces separated by a distance $d$ between their vertices, precisely as was done earlier in Section 5.2.3 where the thin-lens equation was derived. After a great deal of algebraic manipulation,* wherein $d$ is not now negligible, one arrives at a very interesting result for the thick lens immersed in air. The expression for the conjugate points once again can be put in the Gaussian form,

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f},$$

provided that both these object and image distances are measured from the first and second principal planes, respectively. Moreover, the effective focal length or simply the focal length $f$ is also reckoned with respect to the principal planes and is given by

$$\frac{1}{f} = (n_i - 1) \left[ \frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_i - 1)d}{n_1R_1R_2} \right].$$

The principal planes are located at distances of $V_1H_1 = h_1$ and $V_2H_2 = h_2$ which are positive when the planes lie to the right of their respective vertices. Figure 6.4 illustrates the arrangement of the various quantities. The values of $h_1$ and $h_2$ are given by

$$h_1 = -\frac{f(n_i - 1)d}{R_2 n_i},$$

and

$$h_2 = -\frac{f(n_i - 1)d}{R_1 n_i}.$$

In the same way the Newtonian form of the lens equation holds as is evident from the similar triangles in Fig. 6.4. Thus

$$x_i x_o = f^2$$

so long as $f$ is given the present interpretation. And from the same triangles

$$M_T = \frac{y_i}{y_o} = -\frac{x_i}{f} = -\frac{f}{x_o}.$$

Obviously if $d \to 0$, Eqs. (6.1), (6.2), and (6.5) transform into the thin-lens expressions (5.17), (5.16), and (5.23). As a numerical example, let’s find the image distance for an object positioned 30 cm from the vertex of a double convex lens having radii of 20 cm and 40 cm, a thickness of 1 cm and an index of 1.5. From Eq. (6.2) the focal length (in centimeters) is

$$\frac{1}{f} = (1.5 - 1) \left[ \frac{1}{20} - \frac{1}{40} + \frac{(1.5 - 1)1}{1.5(20)(-40)} \right]$$

* For the complete derivation, see Morgan, Introduction to Geometrical and Physical Optics, p. 57.
and so \( f = 26.8 \text{ cm} \). Furthermore
\[ h_1 = \frac{26.8(0.5)1}{-40(1.5)} = +0.22 \text{ cm} \]
and
\[ h_2 = \frac{26.8(0.5)1}{20(1.5)} = -0.44 \text{ cm}, \]
which means that \( H_1 \) is to the right of \( V_1 \) and \( H_2 \) is to the left of \( V_2 \). Finally, \( s_o = 30 + 0.22 \), whence
\[ \frac{1}{30.2} + \frac{1}{s_i} = \frac{1}{26.8}, \]
and \( s_i = 238 \text{ cm} \) measured from \( H_2 \).

The principal points are conjugate to each other. In other words, since \( f = s_o s_i / (s_o + s_i) \), when \( s_o = 0 \), \( s_i \) must be zero because \( f \) is finite and thus a point at \( H_1 \) is imaged at \( H_2 \). Furthermore, an object in the first principal plane \( (x_o = -f) \) is imaged in the second principal plane \( (x_i = -f) \) with unit magnification \( (M_T = 1) \). It is for this reason that these are sometimes spoken of as unit planes. Hence any ray directed toward a point on the first principal plane will emerge from the lens as if it originated at the corresponding point (the same distance above or below the axis) on the second principal plane.

Suppose we now have a compound lens consisting of two thick lenses \( L_1 \) and \( L_2 \) (Fig. 6.5). Let \( s_{o1}, s_{i1}, f_1 \) and \( s_{o2}, s_{i2} \) and \( f_2 \) be the object and image distances and focal lengths for the two lenses, all measured with respect to their own principal planes. We know that the transverse magnification is the product of the magnifications of the individual lenses, that is
\[ M_T = \left( \frac{-s_{i1}}{s_{o1}} \right) \left( \frac{-s_{i2}}{s_{o2}} \right) = \frac{-s_i}{s_o}, \quad (6.7) \]
where \( s_o \) and \( s_i \) are the object and image distances for the combination as a whole. When \( s_o \) is equal to infinity \( s_o = s_{o1}, s_{i1} = f_1, s_{o2} = -(s_{i1} - d) \), and \( s_i = f \). Since
\[ \frac{1}{s_{o2}} + \frac{1}{s_{i2}} = \frac{1}{f} \]
it follows (Problem 6.1), upon substituting into Eq. (6.7), that
\[ \frac{f_1 s_{i2}}{s_{o2}} = f \]

![Fig. 6.4 Thick lens geometry.](image)

![Fig. 6.5 A compound thick lens.](image)
Both are positive and therefore the planes lie to the right of $O_1$ and $O_2$, respectively. Both computed values agree with the results depicted in the diagram. If light enters from the right the system resembles a telephoto lens which must be placed 15 cm from the film plane and yet has an effective focal length of 30 cm.

The same procedures can be extended to three, four, or more lenses. Thus

$$f = f_1 \left( -\frac{s_{12}}{s_{o2}} \right) \left( -\frac{s_{13}}{s_{o3}} \right) \cdots$$  \hspace{2cm} (6.11)

Equivalently, the first two lenses can be envisioned as combined to form a single thick lens whose principal points and focal length are calculated. It, in turn, is combined with the third lens and so on with each successive element.

### 6.2 ANALYTICAL RAY TRACING

Ray tracing is unquestionably one of the designer's chief tools. Having formulated an optical system on paper, he can mathematically shine rays through it to evaluate its performance. Any ray, paraxial or otherwise, can be traced through the system exactly. Conceptually it's a simple matter of applying the refraction equation

$$n_i (k_i \times \hat{u}_h) = n_f (k_f \times \hat{u}_h)$$ \hspace{2cm} [4.7]

at the first surface, locating where the transmitted ray then strikes the second surface; applying the equation once again, and so on all the way through. At one time meridional rays (those in the plane of the optical axis) were traced almost exclusively because non-meridional or skew rays (which do not intersect the axis) are considerably more complicated to deal with mathematically. The distinction is of less importance to a high-speed electronic computer (Fig. 6.7) which simply takes a trifle longer to make the trace. Thus while it would probably take 10 or 15 minutes for a skilled person with a desk calculator to evaluate the trajectory of a single skew ray through a single surface, a computer might require roughly a thousandth of a second for the same job and, equally important, it would be ready for the next calculation with undiminished enthusiasm.

The simplest case which will serve to illustrate the ray-tracing process corresponds to that of a paraxial, meridional
6.2 Analytical ray tracing

6.2.1 Matrix Methods

In the beginning of the nineteen-thirties, T. Smith formulated a rather interesting way of handling the ray-tracing equations. The simple linear form of the expressions and the repetitive manner in which they are applied suggested the use of matrices. The processes of refraction and transfer might then be performed mathematically by matrix operators. The initial insights were not widely appreciated for almost thirty years. However, the early nineteen-sixties saw the beginning of a rebirth of interest which is now flourishing. We shall only outline some of the salient features of the method leaving a more detailed study to the references.

Let's begin by writing the formulas which are not very insightful since we merely replaced $y_1$ in Eq. (6.12) by the symbol $y_{i1}$ and then let $y_{i1} = y_{i2}$. This last bit of business is for purely cosmetic purposes, as you will see in a moment. In effect it simply says that the height of reference point $P_1$ above the axis in the incident medium ($y_{i1}$) equals its height in the transmitting medium ($y_{i2}$)—which is obvious. But now the pair of equations can be expressed as

$$y_2 = y_1 + d_{12} x_{i1}, \quad (6.13)$$

where use was made of the fact that $\tan \alpha_{i1} \approx \alpha_{i1}$. This known as the transfer equation because it allows us to follow the ray from $P_1$ to $P_2$. Recall that the angles are positive if the ray has a positive slope. Since we are dealing with the paraxial region $d_{12} \approx V_{21} V_1$ and $y_2$ is easily computed, Equations (6.11) and (6.12) are then used successively to trace a ray through the entire system. Of course, these are meridional rays and because of the lenses' symmetry about the optical axis such a ray remains in the same meridional plane throughout its sojourn. The process is two-dimensional; there are two equations and two unknowns $x_{i1}$ and $y_{i2}$. In contrast, a skew ray would have to be treated in three dimensions.

recast in matrix form as

\[
\begin{bmatrix}
  n_{11} x_{11} \\
  y_{11}
\end{bmatrix} = \begin{bmatrix} 1 & -d_1 \end{bmatrix} \begin{bmatrix} n_{11} x_{11} \\
  y_{11}
\end{bmatrix}
\]  \hspace{1cm} (6.16)

This could equally well be written as

\[
\begin{bmatrix}
  x_{11} \\
  y_{11}
\end{bmatrix} = \begin{bmatrix} n_{11}/n_{11} & -d_1/n_{11} \end{bmatrix} \begin{bmatrix} x_{11} \\
  y_{11}
\end{bmatrix}
\]  \hspace{1cm} (6.17)

so that the precise form of the 2 \times 1 column matrices is actually a matter of preference. In any case, these can be envisioned as rays on either side of \( P_1 \), one before and the other after refraction. Accordingly, using \( \bullet_1 \) and \( \bullet_1 \) for the two rays, we can write

\[
\begin{bmatrix}
  n_{11} x_{11} \\
  y_{11}
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
  n_{11} x_{11} \\
  y_{11}
\end{bmatrix}
\]  \hspace{1cm} (6.18)

The 2 \times 2 matrix is the refraction matrix denoted as

\[
\mathcal{R}_1 \equiv \begin{bmatrix} 1 & -d_1 \\ 0 & 1 \end{bmatrix}
\]  \hspace{1cm} (6.19)

and so Eq. (6.16) can be concisely stated as

\[
\bullet_1 = \mathcal{R}_1 \bullet_1,
\]  \hspace{1cm} (6.20)

which just says that \( \mathcal{R}_1 \) transforms the ray \( \bullet_1 \) into the ray \( \bullet_1 \) during refraction at the first interface. From Fig. 6.8 we have \( n_{12} x_{12} = n_{11} x_{11} \), that is

\[
n_{12} x_{12} = n_{11} x_{11} + 0
\]  \hspace{1cm} (6.21)

and

\[
y_{12} = d_{21} x_{11} + y_{11}
\]  \hspace{1cm} (6.22)

where \( n_{12} = n_{11}, x_{12} = x_{11} \) and use was made of Eq. (6.13) with \( y_2 \) rewritten as \( y_{12} \) to make things pretty. And thus

\[
\begin{bmatrix}
  n_{12} x_{12} \\
  y_{12}
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d_{21}/n_{11} & 1 \end{bmatrix} \begin{bmatrix} n_{11} x_{11} \\
  y_{11}
\end{bmatrix}
\]  \hspace{1cm} (6.23)

The transfer matrix

\[
\mathcal{F}_{21} \equiv \begin{bmatrix} 1 & 0 \\ d_{21}/n_{11} & 1 \end{bmatrix}
\]  \hspace{1cm} (6.24)

takes the transmitted ray at \( P_1 \), i.e., \( \bullet_1 \), and transforms it into the incident ray at \( P_2 \):

\[
\bullet_{21} \equiv \mathcal{F}_{21} \bullet_1
\]  \hspace{1cm} (6.25)

If we make use of Eq. (6.20) this becomes

\[
\bullet_{12} = \mathcal{F}_{21} \mathcal{R}_1 \bullet_1
\]  \hspace{1cm} (6.26)

The 2 \times 2 matrix formed by the product of the transfer and refraction matrices \( \mathcal{F}_{21} \mathcal{R}_1 \) will carry the ray incident at \( P_1 \) into the ray incident at \( P_2 \). Notice that the determinant of \( \mathcal{F}_{21} \), denoted by \( |\mathcal{F}_{21}| \), equals 1, i.e., (1) \( (1) - (0) (1/n_{11}) = 1 \). Similarly \( |\mathcal{R}_1| = 1 \) and since the determinant of a matrix product equals the product of the individual determinants \( |\mathcal{F}_{21} \mathcal{R}_1| = 1 \). This provides a quick check on the computations. Carrying the procedure through the second interface (Fig. 6.8) of the lens, which has a refraction matrix \( \mathcal{R}_2 \), it follows that
or from Eq. (6.26)
\[ s_{t2} = R_2 s_{t1} \]
(6.27)

The system matrix \( \mathcal{A} \) is defined as
\[ \mathcal{A} = R_2 \mathcal{T}_2 R_1 \]
(6.28)
and has the form
\[ \mathcal{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]
(6.29)

Since
\[ \mathcal{A} = \begin{bmatrix} 1 & -D_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_{21}/n_{11} & 0 \end{bmatrix} \begin{bmatrix} 1 & -D_1 \\ 0 & 1 \end{bmatrix} \]

or
\[ \mathcal{A} = \begin{bmatrix} 1 & -D_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -D_1 \\ d_{21}/n_{11} & -D_1 d_{21}/n_{11} + 1 \end{bmatrix} \]

We can write
\[ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & -D_2 d_{21}/n_{11} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \left( D_2 D_1 d_{21}/n_{11} \right) - D_2 \\ d_{21}/n_{11} & -D_1 d_{21}/n_{11} + 1 \end{bmatrix} \]
(6.31)

and again \( |\mathcal{A}| = 1 \) (Problem 6.5). The values of each element in \( \mathcal{A} \) is expressed in terms of the physical lens parameters such as thickness, index and radii (via \( D \)). Thus the cardinal points which are properties of the lens, determined solely by its make-up, should be deducible from \( \mathcal{A} \). The system matrix in this case (6.31) transforms an incident ray at the first surface to an emerging ray at the second surface; as a reminder we will write it as \( \mathcal{A}_{t1} \).

The concept of image formation enters rather directly (Fig. 6.9) after introducing appropriate object and image planes. Consequently, the first operator \( \mathcal{T}_{10} \) transfers the reference point from the object, i.e. \( P_0 \), to \( P_1 \). The next operator \( \mathcal{A}_{t1} \), then carries the ray through the lens, and a final additional transfer \( \mathcal{T}_{12} \) brings it to the image plane, i.e., \( P_2 \). Thus the ray at the image point \( (s_t) \) is given by
\[ s_t = \mathcal{T}_{12} \mathcal{A}_{t1} \mathcal{T}_{10} s_o \]
(6.32)
where \( s_o \) is the ray at \( P_0 \). In component form this is
\[ \begin{bmatrix} n_0 s_o \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d_{12}/n_1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_{10}/n_0 & 1 \end{bmatrix} \begin{bmatrix} n_0 s_o \\ y_0 \end{bmatrix} \]
(6.33)

Notice that \( \mathcal{T}_{10} s_o = s_1 \) and that \( \mathcal{A}_{t1} s_1 = s_2 \) and hence \( \mathcal{T}_{12} s_2 = s_t \). The subscripts \( 0, 1, 2, \ldots, l \) correspond to reference points \( P_0, P_1, P_2 \) etc. while subscripts \( i \) and \( t \) denote on which side of the reference point we are, i.e., whether incident or transmitted. Operation by a refraction matrix will change \( i \) to \( t \) but not the reference point designation. On the other hand, operation by a transfer matrix obviously does change the latter.

Ordinarily the physical significances of the components of \( \mathcal{A} \) are found by expanding out Eq. (6.33), but this is too involved an approach to do here. Instead, let's return to Eq. (6.31) and examine several of the terms. For example,
\[ -a_{12} = D_1 + D_2 = D_2 D_1 d_{21}/n_{11} \]

If we suppose, for the sake of simplicity, that the lens is in air, then
\[ D_1 = \frac{n_{11} - 1}{R_1} \quad \text{and} \quad D_2 = \frac{n_{11} - 1}{-R_2} \]
as in Eqs. (5.63) and (5.64). Hence
\[ -a_{12} = \left( n_{11} - 1 \right) \left[ \frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_{11} - 1) d_{21}}{R_1 R_2 n_{11}} \right] \]

But this is the expression for the focal length of a thick lens (6.2); in other words
\[ a_{12} = -1/f. \]
(6.34)

If the imbedding media were different on each side of the lens (Fig. 6.10) this would become
\[ a_{12} = -\frac{n_{11}}{f_o} = -\frac{n_{12}}{f_i}. \]
(6.35)
Similarly, it is left as a problem to verify that

$$\frac{V_1 H_1}{R_1} = \frac{n_{i1}(1-a_{ii})}{-a_{i2}} \quad (6.36)$$

and

$$\frac{V_2 H_2}{R_2} = \frac{n_{i2}(a_{j2} - 1)}{-a_{i2}} \quad (6.37)$$

which locate the principal points.

As an example of how the technique can be used, let's apply it, at least in principle, to the Tessar lens* shown in Fig. 6.11. The system matrix has the form

$$A_{1} = R_7 \mathcal{F}_{76} R_6 \mathcal{F}_{65} R_5 \mathcal{F}_{54} R_4 \mathcal{F}_{43} R_3 \mathcal{F}_{32} R_2 \mathcal{F}_{21} R_1,$$

where

$$\mathcal{F}_{21} = \begin{bmatrix} 1 & 0 \\ 0.357 & 1 \\ 1.6116 & 1 \end{bmatrix}, \quad \mathcal{F}_{32} = \begin{bmatrix} 1 & 0 \\ 0.189 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\mathcal{F}_{43} = \begin{bmatrix} 1 & 0 \\ 0.081 & 1 \\ 1.6053 & 1 \end{bmatrix}, \text{ etc.}$$

* We have chosen this particular example primarily because Nussbaum's book *Geometric Optics* contains a simple Fortran computer program specifically written for this lens. It would be almost silly to evaluate the system matrix by hand. Since Fortran is an easily mastered computer language, the program is well worth further study.

Furthermore

$$R_1 = \begin{bmatrix} 1 & -1.6116 & -1 \\ -1.628 & 1 \\ 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & -1.6116 \\ -27.57 & 1 \\ 0 & 1 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 1 & -1.6053 \\ -3.457 & 1 \\ 0 & 1 \end{bmatrix}, \text{ etc.}$$

Multiplying out the matrices, in what is obviously a horrendous although conceptually simple calculation, one presumably will get

$$A_{1} = \begin{bmatrix} 0.848 & -0.198 \\ 1.338 & 0.867 \end{bmatrix}$$

and from that $f = 5.06, V_1 H_1 = 0.77$ and $V_2 H_2 = -0.67$.

As a last point, it is often convenient to consider a system of thin lenses using the matrix representation and to that end return to Eq. (6.31). It describes the system matrix for a single lens and if we let $d_{21} \to 0$, it corresponds to a thin lens. This is equivalent to making $\mathcal{F}_{21}$ a unit matrix, thus
deform the image, as for example, *Petzval field curvature* and *distortion*.

We have known all along that spherical surfaces in general would yield perfect imagery only in the paraxial region. What must now be determined is the kind and extent of the deviations which result simply from using those surfaces with finite apertures. By the judicious manipulation of a system's physical parameters (e.g. the powers, shapes, thicknesses, glass types and separations of the lenses as well as the locations of stops), these aberrations can indeed be minimized. In effect, one cancels out the most undesirable faults by a slight change in the shape of a lens here, or a shift in the position of a stop there (very much like trimming up a circuit with small variable capacitors, coils and pots). When it's all finished, the unwanted deformations of the wavefront incurred as it passes through one surface will hopefully be negated as it traverses some other surfaces further down the line.

Today there are elaborate computer programs for "automatically" doing this kind of analysis. Broadly speaking, you give the computer a quality factor (or merit function) of some sort to aim for, i.e., you essentially tell it how much of each aberration you are willing to tolerate. Then you give it a roughly designed system (e.g. some Tessar configuration) which, in the first approximation, meets the particular requirements. Along with that, you feed in whatever parameters must be held constant, such as a given *f*-number, focal length, or lens diameter, the field of view, or magnification. The computer will then trace several rays through the system and evaluate the image errors. Having been given leave to vary, say, the curvatures and axial separations of the elements, it will calculate the optimum effect of such changes on the quality factor, make them, and then reevaluate. After perhaps twenty or more iterations, usually taking a matter of minutes, it will have changed the initial configuration so that it now meets the specified limits on aberrations. The final lens design will still be a Tessar, but not the one you started with. The result is, if you will, an *optimum configuration* but probably not the optimum. We can be fairly certain that all aberrations cannot be made exactly zero in any real system comprised of spherical surfaces. Moreover, there is no presently known way to determine how close to zero we can actually come. A quality factor is somewhat like a crater-pocked surface in a multidimensional space. The computer will carry the design from one hole to the next until it finds one deep enough to meet the specifications. There it stops and presumably presents us with a perfectly satisfactory configuration. But there is no way to tell if that solution

\[ \mathcal{A} = \mathcal{R}_2 \mathcal{R}_1 = \begin{bmatrix} 1 & -(D_1 + D_2) \\ 0 & 1 \end{bmatrix} \]

But as we saw in Section 5.7.2, the power of a thin lens \( D \) is the sum of the powers of its surfaces. Hence

\[ \mathcal{A} = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/f \end{bmatrix} \]

(6.38)

In addition, for two thin lenses separated by a distance \( d \), in air, the system matrix is

\[ \mathcal{A} = \begin{bmatrix} 1 & -1/f_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/f_1 \\ 0 & 1 \end{bmatrix} \]

or

\[ \mathcal{A} = \begin{bmatrix} 1 - d/f_2 & -1/f_1 + d/f_1 f_2 - 1/f_2 \\ d & -d/f_1 + 1 \end{bmatrix} \]

Clearly then

\[ -a_{12} = \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \]

and from Eqs. (6.36) and (6.37)

\[ O_1 H_1 = f d/f_2, \quad O_2 H_2 = -f d/f_1, \]

all of which by now should be quite familiar. Note how easy it would be with this approach to find the focal length and principal points for a compound lens composed of three, four or more thin lenses.

### 6.3 ABERRATIONS

To be sure, we already know that first-order theory is no more than a good approximation—an exact ray trace or even measurements performed on a prototype system would certainly reveal inconsistencies with the corresponding paraxial description. Such departures from the idealized conditions of Gaussian optics are known as *aberrations*. There are two main classifications of these, namely *chromatic aberrations* (which arise from the fact that \( n \) is actually a function of frequency or color) and *monochromatic aberrations*. The latter occur even with light which is highly monochromatic, and they, in turn, fall into two subgroupings. There are monochromatic aberrations which deteriorate the image making it unclear, such as *spherical aberration*, *coma* and *astigmatism*. In addition, there are aberrations which
corresponds to the deepest hole without sending the computer out again and again meandering along totally different routes.

We mention all of this so that the reader may appreciate the current state of the art. In a word, it is magnificent, but still incomplete; it is "automatic" but a bit myopic.

6.3.1 Monochromatic Aberrations

The paraxial treatment was based on the assumption that \( \sin \varphi \), as in Fig. 5.8, could be represented satisfactorily by \( \varphi \) alone, i.e. the system was restricted to operating in an extremely narrow region about the optical axis. Obviously, if rays from the periphery of a lens are to be included in the formation of an image, the statement that \( \sin \varphi \approx \varphi \) is somewhat unsatisfactory. Recall that we also occasionally wrote Snell's law simply as \( n_i \beta_i = n_o \alpha_o \), which again would be inappropriate. In any event, if the first two terms in the expansion

\[
\sin \varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \cdots
\]

are retained as an improved approximation, we have the so-called third-order theory. Departures from first-order theory which then result are embodied in the five primary aberrations (spherical aberration, coma, astigmatism, field curvature and distortion). These were first studied in detail by Ludwig von Seidel (1821–96) in the eighteen-fifties. Accordingly, they are frequently spoken of as the Seidel aberrations. In addition to the first two contributions, the series obviously contains many other terms, smaller to be sure, but still to be reckoned with. Thus, there are most certainly higher-order aberrations. The difference between the results of exact ray-tracing and the computed primary aberrations can therefore be thought of as the sum of all contributing higher-order aberrations. We shall restrict this discussion to the primary aberrations exclusively.

1) Spherical Aberration

Let's return for a moment to Section 5.2.2 (p.102) where we computed the conjugate points for a single refracting spherical interface. We found then, for the paraxial region, that

\[
\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}.
\]

If the approximations for \( \xi_o \) and \( \xi_i \) are improved a bit (Problem 6.11) we get the third-order expression:

\[
\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2 \left[ \frac{n_1}{2s_o} \left( \frac{1}{s_o} + \frac{1}{R} \right)^2 + \frac{n_2}{2s_i} \left( \frac{1}{s_i} - \frac{1}{R} \right)^2 \right].
\]

The additional term, which varies approximately as \( h^2 \), is clearly a measure of the deviation from first-order theory. As shown in Fig. 6.12 rays striking the surface at greater distances above the axis \( h \) are focused nearer to the vertex. In brief, spherical aberration or SA corresponds to a dependence of focal length on aperture for nonparaxial rays. Similarly, for a converging lens, as in Fig. 6.13, the marginal rays will, in effect, be bent too much, being focused in front of the paraxial rays. The distance between the axial intersection of a ray and the paraxial focus, \( F_1 \), is known as the longitudinal spherical aberration, or L SA. In this case, the SA is positive. In contrast the marginal rays for a diverging lens will generally intersect the axis behind the paraxial focus and we say that its spherical aberration is therefore negative.

If a screen is placed at \( F_1 \) in Fig. 6.13, the image of a star would appear as a bright central spot on the axis surrounded by a symmetrical halo delineated by the cone of marginal rays. For an extended image, SA would reduce the contrast and degrade the details. The height above the axis where a given ray strikes this screen is called the transverse (or lateral) spherical aberration, T SA for short. Evidently, SA can be reduced by stopping down the aperture—but that reduces the amount of light entering the system as well. Notice that if the screen is moved to the position labeled \( \Sigma_{LC} \) the image blur will have its smallest diameter. This is known as the circle of least confusion and \( \Sigma_{LC} \) is generally the best place to observe the image. If a lens exhibits appreciable SA, it
6.3 Aberrations

Fig. 6.13 Spherical aberration for a lens. The envelope of the refracted rays is called a caustic. The intersection of the marginal rays and the caustic locates $\Sigma_{LC}$.

will have to be refocused after it is stopped down because the position of $\Sigma_{LC}$ will approach $F_i$ as the aperture decreases.

The amount of spherical aberration, when the aperture and focal length are fixed, varies with both the object distance and the lens shape. For a converging lens, the non-paraxial rays are too strongly bent. Yet if we imagine the lens as roughly resembling two prisms joined at their bases, it is evident that the incident ray will undergo a minimum deviation when it makes, more or less, the same angle as does the emerging ray (Section 5.5.1). A striking example is illustrated in Fig. 6.14 where simply turning the lens around markedly reduces the SA. When the object is at infinity a simple concave or convex lens which has an almost, but not quite, flat rear side will suffer a minimum amount of spherical aberration. In the same way, if the object and image distances are to be equal ($s_o = s_i = 2f$) the lens should be equiconvex to minimize SA. A combination of a converging and a diverging lens (as in an achromatic doublet) can also be utilized to diminish spherical aberration.

Recall that the aspherical lenses of Section 5.2.1 were completely free of spherical aberration for a specific pair of conjugate points. Moreover, Huygens seems to have been the first to discover that two such axial points exist for spherical surfaces as well. These are shown in Fig. 6.15(a) which depicts rays issuing from $P$ and leaving the surface as if they came from $P'$. It is left for a problem to show that the appropriate locations of $P$ and $P'$ are those indicated in the figure. Just as with the aspherics, lenses can be formed which have this same zero SA for the pair of points $P$ and $P'$. One simply grinds another surface of radius $PA$ centered on $P$ to form either a positive- or negative-meniscus lens. The oil-immersion microscope objective uses this principle to great advantage. The object under study is positioned at $P$ and surrounded by oil of index $n_2$ as in Fig. 6.16. $P$ and $P'$ are the proper conjugate points for zero SA for the first element while $P'$ and $P''$ are those for the meniscus lens.

ii) Coma

Coma or comatic aberration is an image-degrading, monochromatic, primary aberration associated with an object point even a short distance from the axis. Its origins lie in the fact that the principal "planes" can actually be treated as planes only in the paraxial region. They are, in fact, principal curved surfaces (Fig. 6.1). In the absence of SA a parallel bundle of rays will focus at the axial point $F_i$, a distance b.f.l. from the rear vertex. Yet the effective focal lengths and therefore the

Fig. 6.14 SA for a planar-convex lens.
transverse magnifications will differ for rays traversing off-axis regions of the lens. When the image point is on the optical axis, this situation is of little consequence, but when the ray bundle is oblique and the image point is off-axis, coma will be evident. The dependence of $M_T$ on $h$, the ray height at the lens, is evident in Fig. 6.17. Here meridional rays traversing the extremities of the lens arrive at the image plane closer to the axis than do the rays in the vicinity of the principal ray (i.e., the ray which passes through the principal points). In this instance, the least magnification is associated with the marginal rays which would form the smallest image—the coma is said to be negative. By comparison, the coma in Fig. 6.18 is positive because the marginal rays focus further from the axis. Several skew rays are drawn from an extra-axial object point $S$ in Fig. 6.19 to illustrate the formation of the geometrical comatic image of a point. Observe that each circular cone of rays whose endpoints (1-2-3-4-1-2-3-4) form a ring on the lens is imaged in what H. Dennis Taylor called a comatic circle on $\Sigma_i$. This case corresponds to positive coma and so the larger the ring on the lens, the more distant will be its comatic circle from the axis. When the outer ring is the intersection of marginal rays, the distance from 0 to 1 in the image is the tangential coma, while the length from 0 to 3 on $\Sigma_i$ is termed the sagittal coma. A little more than half of the energy in the image appears in the roughly triangular region between 0 and 3. The coma flare, which owes its name to its comet-like tail, is often thought to be the worst of all aberrations, primarily because of its asymmetric configuration.

Like SA, coma is dependent on the shape of the lens. Thus, a strongly concave positive-meniscus lens

![Fig. 6.15](image1.jpg)

![Fig. 6.16](image2.jpg)

- Corresponding axial points for which SA is zero.
- An oil-immersion microscope objective.
the object at infinity will have a large negative coma. Bending the lens so that it becomes planar-convex \( \subseteq \), then equi-convex \( \subseteq \), convex-planar \( \subseteq \), and finally convex-meniscus \( \subseteq \), will change the coma from negative, to zero, to positive. The fact that it can be made exactly zero for a single lens with a given object distance is quite significant. The particular shape it would then have \( (s_\infty = \infty) \) is almost convex-planar and very nearly the configuration for minimum SA.

It is quite important to realize that a lens which is well corrected for the case where one conjugate point is at infinity \( (s_\infty = \infty) \) may not perform satisfactorily when the object is nearby. One would therefore do well, when using off-the-shelf lenses in a system operating at finite conjugates, to combine two infinite conjugate corrected lenses as in Fig. 6.20. In other words, since it is unlikely that a lens with the desired focal length, which is also corrected for the particular set of finite conjugates, can be obtained ready made, this back-to-back lens approach is an appealing alternative.

Coma can also be negated by using a stop at the proper location, as was discovered in 1812 by William Hyde Wollaston (1766–1828). The order of the list of primary aberrations \( (\text{SA, coma, astigmatism, Petzval field curvature and distortion}) \) is significant because any one of them, except SA and Petzval curvature, will be affected by the position of a stop, but only if one of the preceding aberrations is already present in the system. Thus while SA is independent of the location along the axis of a stop, coma will not be, as long as SA is present. This can better be appreciated by examining the representation in Fig. 6.21. With the stop at \( \Sigma_1 \), ray 3 is the chief ray, there is SA but no coma; i.e., the ray pairs meet on 3. If the stop is moved to \( \Sigma_2 \), the symmetry is upset, ray 4 becomes the chief ray and the rays on either side of it, such as 3 and 5, meet above it, not on it—there is then positive coma. With the stop at \( \Sigma_3 \), the rays 1 and 3 intersect below the chief ray, 2, and there is negative coma. In this way, controlled amounts of the aberration can be introduced into a compound lens in order to cancel coma in the system as a whole.

The \textit{optical sine theorem} is an important relationship which must be introduced here even if space precludes its formal proof. It was discovered independently in 1873 by Abbe and Helmholtz although a different form of it was given ten years earlier by R. Clausius (of thermodynamics fame). In any event, it states that

\[
 n_0 y_o \sin \alpha_o = n_1 y_1 \sin \alpha_1, \tag{6.41}
\]
More on geometrical optics

6.3

(6.42) \[ \frac{\sin \alpha_i}{\sin \alpha_p} = \frac{x_i}{x_p} = \text{constant}, \]

must be constant for all rays. Suppose then that we send a marginal and a paraxial ray through the system. The former will comply with Eq. (6.41), the latter with its paraxial version (in which \( \sin \alpha_i = \alpha_{op}, \sin \alpha_p = \alpha_{ip} \)). Since \( M_T \) is to be constant over the entire lens, we equate the magnification for both marginal and paraxial rays to get

\[ M_T = \frac{v_i}{y_o} \] \[ (5.24) \]

which is known as the *sine condition*. A necessary criterion for the absence of coma is that the system meet the sine condition. If there is no SA, compliancy with the sine condition will be both necessary and sufficient for zero coma.

It's an easy matter to observe coma. In fact, anyone who has focused sunlight with a simple positive lens has no doubt seen the effects of this aberration. A slight tilt of the lens, so that the nearly collimated rays from the sun make an angle with the optical axis, will cause the focused spot to flare out into the characteristic comet shape.

**iii) Astigmatism**

When an object point lies an appreciable distance from the optical axis the incident cone of rays will strike the lens asymmetrically, giving rise to a third primary aberration known as *astigmatism*. To facilitate its description, envision the meridional plane (also called the *tangential plane*) containing both the chief ray (i.e. the one passing through the center of the aperture) and the optical axis. The *sagittal plane* is then defined as the plane containing the chief ray which, in addition, is perpendicular to the meridional plane (Fig. 6.22). Unlike the latter which is unbroken from one end of a complicated lens system to the other, the sagittal plane generally changes slope as the chief ray is deviated at

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* To be precise, the sine theorem is valid for all values of \( \alpha \), only in the sagittal plane (from the Latin *sagitta* meaning arrow) which is discussed in the next section.
the various elements. Hence to be accurate we should say that there are actually several sagittal planes, one attendant with each region within the system. Nevertheless, all skew rays from the object point lying in a sagittal plane are termed **sagittal rays**.

In the case of an axial object point, the cone of rays is symmetrical with respect to the spherical surfaces of a lens. There is no need to make a distinction between meridional and sagittal planes. The ray configurations in all planes containing the optical axis are identical. In the absence of spherical aberration, all the focal lengths are the same and consequently all rays arrive at a single focus. Contrastingly, the configuration of an oblique, parallel ray bundle will be different in the meridional and sagittal planes. As a result, the focal lengths in these planes will be different as well. In effect, here the meridional rays are tilted more with respect to the lens than are the sagittal rays and they have a shorter focal length. It can be shown,* using Fermat's principle, that the *focal length difference* depends effectively on the power of the lens (as opposed to the shape or index) and the angle at which the rays are inclined. This *astigmatic difference*, as it is often called, increases rapidly as the rays become more oblique, i.e. as the object point moves further off the axis and is, of course, zero on axis.

Having two distinct focal lengths, the incident conical bundle of rays takes on a considerably altered form after refraction (Fig. 6.23). The cross-section of the beam as it leaves the lens is initially circular but it gradually becomes elliptical with the major axis in the sagittal plane until at the

$tangential$ or $meridional$ focus $F_T$, the ellipse degenerates into a *line* (at least in third-order theory). All rays from the object point traverse this line which is known as the *primary image*. Beyond this point the beam's cross-section rapidly opens out until it is again circular. At that location the image is a circular blur known as the *circle of least confusion*. Moving further from the lens the beam's cross-section again deforms into a line called the *secondary image*. This time it's in the meridional plane at the *sagittal focus* $F_S$. Remember that all of this is assuming the absence of SA and coma.

Since the circle of least confusion increases in diameter as the astigmatic difference increases, i.e. as the object moves further off-axis, the image will deteriorate, losing definition around its edges. Observe that the secondary line image will change in orientation with changes in the object position but it will always point toward the optical axis, i.e. it will be

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* See A. W. Barton, A Text Book on Light, p. 124.
radial. Similarly, the primary line image will vary in orientation but it will remain normal to the secondary image. This arrangement causes the interesting effect shown in Fig. 6.24 when the object is made up of radial and tangential elements. The primary and secondary images are, in effect, formed of transverse and radial dashes which increase in size with distance from the axis. In the latter case, the dashes point like arrows toward the center of the image—ergo, the name sagitta.

The existence of the sagittal and tangential foci can be verified directly with a fairly simple arrangement. Place a short focal length positive lens (say about 10 or 20 mm) in the beam of an He-Ne laser. Position another, somewhat longer focal length, positive test lens far enough away so that the now diverging beam fills that lens. A convenient object, to be located between the two lenses, is a piece of ordinary wire screening (or a transparency). Align it so the wires are horizontal \((x)\) and vertical \((y)\). If the test lens is rotated through roughly 45° about the vertical (with the \(x\)-, \(y\)-, \(z\)-axes fixed in the lens), astigmatism should be observable. The meridional is the \(xz\)-plane (\(z\) being the lens axis, now at about 45° to the laser axis) while the sagittal plane corresponds to the plane of \(y\) and the laser axis. As the wire mesh is moved toward the test lens, a point will be reached where the horizontal wires are in focus on a screen beyond the lens, while the vertical wires are not. This is the location of the sagittal focus. Each point on the object is imaged as a short line in the meridional (horizontal) plane, which accounts for the fact that only the horizontal wires are in focus. Moving the mesh slightly closer to the lens will bring the vertical lines into clarity while the horizontal ones are blurred. This is the tangential focus. Try rotating the mesh about the central laser axis while at either focus.

Note that unlike visual astigmatism which arose from an actual asymmetry in the surfaces of the optical system, the third-order aberration by that same name applies to spherically symmetric lenses.

Mirrors, with the singular exception of the plane mirror, suffer much the same monochromatic aberrations as do lenses. Thus while a paraboloidal mirror is free of SA for an infinitely distant axial object point, its off-axis imagery is quite poor due to astigmatism and coma. This strongly restricts its use to narrow field devices such as searchlights and astronomical telescopes. A concave spherical mirror shows SA, coma and astigmatism. Indeed one could draw a diagram just like Fig. 6.23 with the lens replaced by an obliquely illuminated spherical mirror. Incidentally, such a mirror displays appreciably less SA than would a simple convex lens of the same focal length.

iv) Field Curvature

Suppose that we have an optical system which is free of all of the aberrations thus far considered. There would then be a one-to-one correspondence between points on the object and image surfaces (i.e. stigmatic imagery). We mentioned earlier [Section 5.2.3(iii)] that a planar object normal to the axis will be imaged approximately as a plane only in the
paraxial region. At finite apertures the resulting curved stigmatic image surface is a manifestation of the primary aberration known as Petzval field curvature after the Hungarian mathematician Josef Max Petzval (1807–91). The effect can readily be appreciated by examining Figs. 5.22 (p. 111) and 6.25. A spherical object segment \( \sigma_o \) is imaged by the lens as a spherical segment \( \sigma'_o \), both centered at \( O \). Flattening out \( \sigma_o \) into the plane \( \sigma'_o \) will cause each object point to move toward the lens along the concomitant chief ray, thus forming a paraboloidal Petzval surface \( \Sigma_p \). While the Petzval surface for a positive lens curves inward toward the object plane, for a negative lens it curves outward, i.e. away from that plane. Evidently, a suitable combination of positive and negative lenses will negate field curvature. Indeed, the displacement \( \Delta x \) of an image point at height \( y_i \) on the Petzval surface from the paraxial image plane is given by

\[
\Delta x = \frac{y_i^2}{2} \sum_{j=1}^{m} \frac{1}{n_j f_j},
\]

where \( n_j \) and \( f_j \) are the indices and focal lengths of the \( m \) thin lenses forming the system. This implies that the Petzval surface will be unaltered by changes in the positions or shapes of the lenses, or in the location of the stop, so long as the values of \( n_j \) and \( f_j \) are fixed. Notice that for the simple case of two thin lenses \( (m = 2) \) having any spacing, \( \Delta x \) can be made zero provided that

\[
\frac{1}{n_1 f_1} + \frac{1}{n_2 f_2} = 0
\]

or equivalently

\[
n_1 f_1 + n_2 f_2 = 0.
\]

This is the so-called Petzval condition. As an example of its use, suppose we combine two thin lenses, one positive, the other negative, such that

\[
f = f_1 = -f_2
\]

and \( n_1 = n_2 \). Since

\[
\frac{1}{f_1} + \frac{1}{f_2} = \frac{d}{f_1 f_2}
\]

the system can satisfy the Petzval condition, have a flat field, and still have a finite positive focal length.

In visual instruments a certain amount of curvature can be tolerated because the eye can accommodate for it. Clearly in photographic lenses field curvature is most undesirable since it has the effect of rapidly blurring the off-axis image when the film plane is at \( F_1 \). An effective means of nullifying the inward curvature of a positive lens is to place a negative field flattener lens near the focal plane. This is often done in projection and photographic objectives when it is not otherwise practicable to meet the Petzval condition (Fig. 6.26). In this position the flattener will have little effect on other aberrations (take another look at Fig. 6.7).

Astigmatism is intimately related to field curvature. In the presence of the former aberration, there will be two paraboloidal image surfaces, the tangential, \( \Sigma_T \), and the sagittal, \( \Sigma_S \) (as in Fig. 6.27). These are the loci of all the
primary and secondary images, respectively, as the object point roams over the object plane. At a given height \(y_i\), a point on \(\Sigma_T\) always lies three times as far from \(\Sigma_R\) as does the corresponding point on \(\Sigma_S\) and both are on the same side of the Petzval surface (Fig. 6.27). When there is no astigmatism \(\Sigma_S\) and \(\Sigma_T\) coalesce on \(\Sigma_R\). It is possible to alter the shapes of \(\Sigma_S\) and \(\Sigma_T\) by bending or relocating the lenses or by moving the stop. The configuration of Fig. 6.27(b) is known as an \textit{artificially flattened} field. A stop in front of an inexpensive meniscus box camera lens is usually arranged to produce just this effect. The surface of least confusion, \(\Sigma_{LC}\), is planar and the image there is tolerable, losing definition at the margins due to theastigmatism. That is to say, although their loci form \(\Sigma_{LC}\), the circles of least confusion increase in diameter with distance off the axis. Modern good-quality photographic objectives are generally \textit{anastigmas}, i.e. they are designed so that \(\Sigma_S\) and \(\Sigma_T\) cross each other yielding an additional off-axis angle of zero astigmatism. The Cooke Triplet, Tessar, Orthometer and Biotar (Fig. 5.100) are all anastigmas as is the relatively fast Zeiss Sonnar whose residual astigmatism is illustrated graphically in Fig. 6.28. Note the relatively flat field and small amount of astigmatism over most of the film plane.

Let’s return briefly to the Schmidt camera of Fig. 5.95 (p. 157) since we are now in a better position to appreciate how it functions. With a stop at the center of curvature of the spherical mirror, all chief rays, which by definition pass through \(C\), are incident normally on the mirror. Moreover each pencil of rays from a distant object point is symmetrical about its chief ray. In effect, each chief ray serves as an optical axis and so there are no off-axis points and in principle no coma or astigmatism. Instead of attempting to flatten the image surface, curvature is coped with by simply shaping the film to conform with it.

\v Distortion

The last of the five primary, monochromatic aberrations is \textit{distortion}. Its origin lies in the fact that the transverse magnification, \(M_T\), may be a function of the off-axis image distance, \(y_i\). Thus, that distance may differ from the one predicted by paraxial theory in which \(M_T\) is constant. In the absence of any of the others, this aberration is manifest in a misshaping of the image as a whole, even though each point is sharply focused. Consequently, when processed by an optical system suffering positive or \textit{pin-cushion distortion}, a square array deforms as in Fig. 6.29(b). In that instance, each image point is displaced radially outward from the center with the most distant points moving the greatest amount, i.e., \(M_T\) increases with \(y_i\). Similarly \textit{negative or barrel distortion} corresponds to the situation where \(M_T\) decreases with the axial distance and, in effect, each point on the image moves radially inward toward the center [Fig. 6.29(c)]. Distortion can easily be seen by just looking through an aberrant lens at a piece of lined or graph paper. Fairly thin lenses will show essentially no distortion whereas ordinary positive or negative, thick, simple lenses will generally suffer positive or negative distortion, respectively.
The introduction of a stop into a system of thin lenses is invariably accompanied by distortion, as indicated in Fig. 6.30. One exception to this is when the aperture stop is at the lens so that the chief ray is, in effect, the principal ray (i.e. it passes through the principal points, here coalesced at O). If the stop is in front of a positive lens, as in Fig. 6.30(b), the object distance measured along the chief ray will be greater than it was with the stop at the lens ($S_2A > S_2O$). Thus $x_o$ will be greater and (5.26) $M_T$ will be smaller—ergo, barrel distortion. In other words, $M_T$ for an off-axis point will be less with a front stop in position than it would be without it. The difference is a measure of the aberration which, by the way, exists regardless of the size of the aperture. In the same way a rear stop [Fig. 6.30(c)] decreases $x_o$ along the chief ray, (i.e., $S_2O > S_2B$), thereby increasing $M_T$ and introducing pin-cushion distortion. Interchanging the object and image thus has the effect of changing the sign of the distortion for a given lens and stop. The aforementioned stop positions will produce the opposite effect when the lens is negative.

All of this rather suggests using a stop midway between identical lens elements. The distortion from the first lens

![Fig. 6.27 The tangential, sagittal and Petzval image surfaces.](image)

![Fig. 6.28 A typical Sonnar. The markings C, S, and E denote the limits of the 35 mm film format (field stop) i.e. corners, sides and edges. The Sonnar family lies between the double Gauss and the triplet.](image)
would precisely cancel the contribution from the second. This approach has been used to advantage in the design of a number of photographic lenses (Fig. 5.100). To be sure, if the lens is perfectly symmetrical and operating as in Fig. 6.30(d) the object and image distances will be equal and hence $M_T = 1$. (Incidentally, coma and lateral color would then be identically zero as well.) This applies to (finite conjugate) copy lenses used, for example, to record data. Nonetheless, even when $M_T$ is not one, making the system approximately symmetrical about a stop is a very common practice since it markedly reduces these several aberrations.

Distortion can arise in compound lens systems, as for example in the telephoto arrangement shown in Fig. 6.31. For a distant object point, the margin of the positive achromat serves as the aperture stop. In effect, the arrangement is like a negative lens with a front stop and so it displays positive or pin-cushion distortion.

Suppose a chief ray enters and emerges from an optical system in the same direction as e.g. in Fig. 6.30(d). The point at which the ray crosses the axis is the optical center of the system; but at the same time, since this is a chief ray, it is also the center of the aperture stop. This is the situation approached in Fig. 6.30(a) with the stop up against the thin lens. In both instances the incoming and outgoing segments of the chief ray are parallel and there is zero distortion, i.e. the system is orthoscopic. This also implies that the entrance and exit pupils will correspond to the principal planes (if the system is immersed in a single medium—see Fig. 6.2). Bear in mind that the chief ray is now a principal ray. A thin-lens system will have zero distortion if its optical center is coincident with the center of the aperture stop. By the way, in a pinhole camera, the rays connecting conjugate object and image points are straight and pass through the center of the aperture stop. The entering and emerging rays are obviously parallel (being one and the same) and there is no distortion.

### 6.3.2 Chromatic Aberrations

The five primary or Seidel aberrations have been considered in terms of monochromatic light. To be sure, if the source had a broad spectral bandwidth these aberrations would be
influenced accordingly; but the effects are inconsequential unless the system is quite well corrected. There are, however, chromatic aberrations that arise specifically in polychromatic light which, by comparison, are far more significant. The ray-tracing equation (6.12) is a function of the indices of refraction which, in turn, vary with wavelength. Different "colored" rays will traverse a system along different paths, and this is the quintessential feature of chromatic aberration.

Since the thin-lens equation

\[
\frac{1}{f} = (n_i - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]

is wavelength dependent via \( n_i(\lambda) \), the focal length must also vary with \( \lambda \). In general (Fig. 3.13, p. 42) \( n_i(\lambda) \) decreases with wavelength over the visible region and thus \( f(\lambda) \) increases with \( \lambda \). The result is illustrated in Fig. 6.32 where the constituent colors in a collimated beam of white light are focused at different points on the axis. The axial distance between two such focal points spanning a given frequency range (e.g. blue to red) is termed the axial (or longitudinal) chromatic aberration, A \cdot CA for short.

It's an easy matter to observe chromatic aberrations, or CA, with a thick, simple converging lens. When illuminated by a polychromatic point source (a candle flame will do) the lens will cast a real image surrounded by a halo. If the plane of observation is then moved nearer the lens, the periphery of the blurred image will become tinged in orange-red. Moving it back away from the lens, beyond the best image, will cause the outlines to become tinted in blue-violet. The location of the circle of least confusion (i.e. the plane \( \Sigma_{LC} \)) corresponds to the position where the best image will appear. Try looking directly through the lens at a source—the coloration will be far more striking.

The image of an off-axis point will be formed of the constituent frequency components, each arriving at a different height above the axis (Fig. 6.33). In essence, the frequency dependence of \( f \) causes a frequency dependence of the transverse magnification as well. The vertical distance between two such image points (most often taken to be blue and red) is a measure of the lateral chromatic aberration L \cdot CA, or lateral color. Consequently, a chromatically aberrant lens illuminated by white light will fill a volume of space with a continuum of more or less overlapping images, varying in size and color. Because the eye is most sensitive to the yellow-green portion of the spectrum, the tendency is to focus the lens for that region. With such a configuration one would see all of the other colored images superimposed and slightly out of focus producing a whitish blur or hazed overlay.

When the blue focus, \( F_B \), is to the left of the red focus, \( F_R \), the A \cdot CA is said to be positive, as it is in Fig. 6.32. Contrarily, a negative lens would generate negative A \cdot CA, with the more strongly deviated blue rays appearing to originate at the right of the red focus. Physically what is happening is that the lens, whether convex or concave, is prismatic in shape, i.e. it becomes either thinner or thicker as the radial distance from the axis increases. As you well know, rays are therefore deviated either toward or away from the axis, respectively. In both cases the rays are bent toward the thicker "base" of the prismatic cross section. But the angular deviation is an increasing function of \( n \) and therefore it decreases with \( \lambda \). Hence blue light is deviated the most and is focused nearest the lens. In other words, for a convex lens the red focus is furthest and to the right; for a concave lens it is furthest and to the left.

i) Thin Achromatic Doublets

All of this rather suggests that a combination of two thin lenses, one positive and one negative, could conceivably result in the precise overlapping of \( F_R \) and \( F_B \) (Fig. 6.34).
Such an arrangement is said to be achromatized for those two specific wavelengths. Notice that what we would like to do is effectively eliminate the total dispersion (i.e. the fact that each color is deviated by a different amount) and not the total deviation itself. With the two lenses separated by a distance $d$,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}. \quad [6.8]$$

Rather than carrying around the second term in the thin-lens equation (5.16) let's abbreviate the notation and write $1/f_1 = (n_1 - 1)\rho_1$ and $1/f_2 = (n_2 - 1)\rho_2$ for the two elements. Then

$$\frac{1}{f} = (n_1 - 1)\rho_1 + (n_2 - 1)\rho_2 - d(n_1 - 1)\rho_1(n_2 - 1)\rho_2. \quad (6.45)$$

This expression will yield the focal length of the doublet for red ($f_R$) and blue ($f_B$) light when the appropriate indices are introduced, namely $n_{1R}$, $n_{2R}$, $n_{1B}$, and $n_{2B}$. But if $f_R$ is to equal $f_B$, then

$$1/f_R = 1/f_B$$

and

$$(n_{1R} - 1)\rho_1 + (n_{2R} - 1)\rho_2 - d(n_{1R} - 1)\rho_1(n_{2R} - 1)\rho_2$$

$$= (n_{1B} - 1)\rho_1 + (n_{2B} - 1)\rho_2 - d(n_{1B} - 1)\rho_1(n_{2B} - 1)\rho_2.$$  

(6.46)

One case of particular importance corresponds to $d = 0$, i.e. the two lenses are in contact. Expanding out Eq. (6.46) with $d = 0$ then leads to

$$\frac{\rho_1}{\rho_2} = \frac{n_{2B} - n_{2R}}{n_{1B} - n_{1R}}. \quad (6.47)$$

The focal length of the compound lens ($f_Y$) can conveniently be specified as that associated with yellow light, roughly midway between the blue and red extremes. For the component lenses in yellow light $1/f_{1Y} = (n_{1Y} - 1)\rho_1$ and $1/f_{2Y} = (n_{2Y} - 1)\rho_2$. Hence

$$\frac{\rho_1}{\rho_2} = \frac{(n_{2Y} - 1) f_{2Y}}{(n_{1Y} - 1) f_{1Y}}. \quad (6.48)$$

Equating Eqs. (6.47) and (6.48) leads to

$$\frac{f_{2Y}}{f_{1Y}} = -\frac{(n_{2B} - n_{2R})/(n_{2Y} - 1)}{(n_{1B} - n_{1R})/(n_{1Y} - 1)}. \quad (6.49)$$

The quantities

$$\frac{n_{2B} - n_{2R}}{n_{2Y} - 1} \quad \text{and} \quad \frac{n_{1B} - n_{1R}}{n_{1Y} - 1}$$

are known as the dispersive powers of the two materials forming the lenses. Their reciprocals, $V_2$ and $V_1$, are variously known as the dispersive indices, V-numbers, or Abbe numbers. Thus

$$\frac{f_{2Y}}{f_{1Y}} = -\frac{V_1}{V_2}$$

or

$$f_{1Y}V_1 + f_{2Y}V_2 = 0. \quad (6.50)$$

Since the dispersive powers are positive, so too are the V-numbers. This implies, as we anticipated, that one of the two component lenses must be negative and the other positive if Eq. (6.50) is to obtain, i.e. if $f_R$ is to equal $f_B$.

At this point we could presumably design an achromatic doublet and indeed we presently shall, but a few additional points must be made first. The designation of wavelengths as red, yellow, and blue is far too imprecise for practical application. Instead it is customary to refer to specific spectral lines whose wavelengths are known with great precision. The Fraunhofer lines, as they are called, serve as the needed reference markers across the spectrum. Several of these for the visible region are listed in Table 6.1. The lines $F$, $C$, and $d$ (i.e. $D_3$) are most often used (for blue, red, and yellow) and
one generally traces paraxial rays in \( d \)-light. Glass manufacturers will usually list their wares in terms of the Abbe number as in Fig. 6.35 which is a plot of the refractive index versus

\[
V_d = \frac{n_d - 1}{n_F - n_C}. \tag{6.51}
\]

(Take a look at Table 6.2 as well.) Thus Eq. (6.50) might better be written as

\[
f_{1d}V_{1d} + f_{2d}V_{2d} = 0, \tag{6.52}
\]

where the numerical subscripts pertain to the two glasses used in the doublet and the letter relates to the \( d \)-line.

Incidentally, Newton erroneously concluded, on the basis of experiments with the very limited range of materials available at the time, that the dispersive power was constant for all glasses. This is tantamount to saying (6.52) that

\[
f_{1d} = -f_{2d},
\]

in which case the doublet would have zero power. Newton, accordingly, shifted his efforts from the refracting to the reflecting telescope and this fortunately turned out to be a good move in the long run. The achromat
was invented around 1733 by Chester Moor Hall, Esq., but it lay in limbo until it was seemingly reinvented and patented in 1758 by the London optician John Dollond.

### Table 6.1 Several strong Fraunhofer lines.

<table>
<thead>
<tr>
<th>Designation</th>
<th>Wavelength (Å)</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>6562.816 red</td>
<td>H</td>
</tr>
<tr>
<td>$D_1$</td>
<td>5895.923 yellow</td>
<td>Na</td>
</tr>
<tr>
<td>$D_2$</td>
<td>5892.9 yellow</td>
<td>Na</td>
</tr>
<tr>
<td>$D_3$ or $d$</td>
<td>5875.618 green</td>
<td>He</td>
</tr>
<tr>
<td>$b_1$</td>
<td>5183.618 green</td>
<td>Mg</td>
</tr>
<tr>
<td>$b_2$</td>
<td>5172.699 green</td>
<td>Mg</td>
</tr>
<tr>
<td>$c$</td>
<td>4957.609 green</td>
<td>Fe</td>
</tr>
<tr>
<td>$F$</td>
<td>4861.327 blue</td>
<td>H</td>
</tr>
<tr>
<td>$f$</td>
<td>4340.465 violet</td>
<td>H</td>
</tr>
<tr>
<td>$g$</td>
<td>4226.728 violet</td>
<td>Ca</td>
</tr>
<tr>
<td>$K$</td>
<td>3933.666 violet</td>
<td>Ca</td>
</tr>
</tbody>
</table>

Several forms of the achromatic doublet are shown in Fig. 6.36. Their configurations depend on the glass types selected as well as on the choice of the other aberrations to be controlled. By the way, when purchasing off-the-shelf doublets of unknown origin, be careful not to buy a lens which deliberately includes certain aberrations in order to compensate for errors in the original system from which it came. Perhaps the most commonly encountered doublet is the cemented Fraunhofer achromat. It’s formed of a crown* double-convex lens in contact with a concave-planar (or nearly planar) flint lens. The use of a crown front element is quite popular because of its better wear resistance. Since the overall shape is roughly convex-planar, by selecting the proper glasses both spherical aberration and coma can be corrected as well. Suppose then, that we wish to design a Fraunhofer achromat of focal length 50 cm. We can get some idea of roughly how to select glasses by solving Eq. (6.52) simultaneously with the compound-lens equation

\[
\frac{1}{f_{1d}} + \frac{1}{f_{2d}} = \frac{1}{f_d}
\]

to get

\[
\frac{1}{f_{1d}} = \frac{V_{1d}}{f_d (V_{1d} - V_{2d})}
\]  

\[(6.53)\]

and

\[
\frac{1}{f_{2d}} = \frac{V_{2d}}{f_d (V_{1d} - V_{2d})}
\]

\[(6.54)\]

Thus, in order to avoid small values of $f_{1d}$ and $f_{2d}$, which would necessitate strongly curved surfaces on the component lenses, the difference $V_{1d} - V_{2d}$ should be made large (roughly 20 or more is convenient). From Fig. 6.35 (or its equivalent) we select, say, BK1 and F2. These have cataloged indices of $n_C = 1.50763$, $n_d = 1.51009$, $n_F = 1.51566$, and $n_C = 1.61503$, $n_d = 1.62004$, and $n_F = 1.63208$, respectively. Likewise their V-numbers are generally given rather accurately and we needn’t compute them. In this instance they are

### Table 6.2 Optical glass.

<table>
<thead>
<tr>
<th>Type number</th>
<th>Name</th>
<th>$n_d$</th>
<th>$V_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>511:635</td>
<td>Borosilicate crown—BSC-1</td>
<td>1.5110</td>
<td>63.5</td>
</tr>
<tr>
<td>517:645</td>
<td>Borosilicate crown—BSC-2</td>
<td>1.5170</td>
<td>64.5</td>
</tr>
<tr>
<td>513:605</td>
<td>Crown—C</td>
<td>1.5125</td>
<td>60.5</td>
</tr>
<tr>
<td>518:596</td>
<td>Crown</td>
<td>1.5180</td>
<td>59.6</td>
</tr>
<tr>
<td>523:586</td>
<td>Crown—C-1</td>
<td>1.5230</td>
<td>58.6</td>
</tr>
<tr>
<td>529:516</td>
<td>Crown flint—CF-1</td>
<td>1.5286</td>
<td>51.6</td>
</tr>
<tr>
<td>541:599</td>
<td>Light barium crown—LBC-1</td>
<td>1.5411</td>
<td>59.9</td>
</tr>
<tr>
<td>573:574</td>
<td>Barium crown—LBC-2</td>
<td>1.5725</td>
<td>57.4</td>
</tr>
<tr>
<td>574:577</td>
<td>Barium crown</td>
<td>1.5744</td>
<td>57.7</td>
</tr>
<tr>
<td>611:588</td>
<td>Dense barium crown—DBC-1</td>
<td>1.6110</td>
<td>58.8</td>
</tr>
<tr>
<td>617:550</td>
<td>Dense barium crown—DBC-2</td>
<td>1.6170</td>
<td>55.0</td>
</tr>
<tr>
<td>611:572</td>
<td>Dense barium crown—DBC-3</td>
<td>1.6109</td>
<td>57.2</td>
</tr>
<tr>
<td>562:510</td>
<td>Light barium flint—LBF-2</td>
<td>1.5616</td>
<td>51.0</td>
</tr>
<tr>
<td>588:534</td>
<td>Light barium flint—LBF-1</td>
<td>1.5880</td>
<td>53.4</td>
</tr>
<tr>
<td>584:460</td>
<td>Barium flint—BF-1</td>
<td>1.5838</td>
<td>46.0</td>
</tr>
<tr>
<td>605:436</td>
<td>Barium flint—BF-2</td>
<td>1.6053</td>
<td>43.6</td>
</tr>
<tr>
<td>559:452</td>
<td>Extra light flint—ELF-1</td>
<td>1.5585</td>
<td>45.2</td>
</tr>
<tr>
<td>573:425</td>
<td>Light flint—LF-1</td>
<td>1.5725</td>
<td>42.5</td>
</tr>
<tr>
<td>580:410</td>
<td>Light flint—LF-2</td>
<td>1.5795</td>
<td>41.0</td>
</tr>
<tr>
<td>605:380</td>
<td>Dense flint—DF-1</td>
<td>1.6050</td>
<td>38.0</td>
</tr>
<tr>
<td>617:366</td>
<td>Dense flint—DF-2</td>
<td>1.6170</td>
<td>36.6</td>
</tr>
<tr>
<td>621:362</td>
<td>Dense flint—DF-3</td>
<td>1.6210</td>
<td>36.2</td>
</tr>
<tr>
<td>649:338</td>
<td>Extra dense flint—EDF-1</td>
<td>1.6490</td>
<td>33.8</td>
</tr>
<tr>
<td>666:324</td>
<td>Extra dense flint—EDF-5</td>
<td>1.6660</td>
<td>32.4</td>
</tr>
<tr>
<td>673:322</td>
<td>Extra dense flint—EDF-2</td>
<td>1.6725</td>
<td>32.2</td>
</tr>
<tr>
<td>689:309</td>
<td>Extra dense flint—EDF</td>
<td>1.6890</td>
<td>30.9</td>
</tr>
<tr>
<td>720:293</td>
<td>Extra dense flint—EDF-3</td>
<td>1.7200</td>
<td>29.3</td>
</tr>
</tbody>
</table>

* Traditionally the glasses in the range $n_d > 1.60$, $V_d > 50$, and $n_f < 1.60$, $V_d > 55$ are known as crowns while the others are flints. Note the letter designations of Fig. 6.35.


Type number is given by $(n_d - 1) : (10 V_d)$ where $n_d$ is rounded off to three decimal places—For more data see Smith, Modern Optical Engineering, Fig. 7.5.
Aberrations

6.3

Element are $R_{II} = 21.8$ cm and $R_{12} = -21.8$ cm while the flint has radii of $R_{21} = -21.8$ cm and $R_{22} = -381.9$ cm.

Note that for a thin-lens combination the principal planes coalesce so that achromatizing the focal length corrects both $A \cdot CA$ and $L \cdot CA$. In a thick doublet, however, even though the focal lengths for red and blue are made identical, the different wavelengths may have different principal planes. Consequently, although the magnification is the same for all wavelengths, the focal points may not coincide, i.e. $L \cdot CA$ is corrected for but not $A \cdot CA$.

In the above analysis only the $C$- and $F$-rays were brought to a common focus while the $d$-line was introduced to establish a focal length for the doublet as a whole. It is not possible for all wavelengths traversing a doublet achromat to meet at a common focus. The resulting residual chromatism is known as secondary spectrum. The elimination of secondary spectrum is particularly troublesome when the design is limited to the glasses presently available. Notwithstanding that, a fluorite ($CaF_2$) element combined with an appropriate glass element can form a doublet achromatized at three wavelengths and having very little secondary spectrum. More often triplets are used for color correction at three or even four wavelengths. The secondary spectrum of a binocular can easily be observed by looking at a distant white object. Its borders will be slightly haloed in magenta and green—try shifting the focus forward and backward.

ii) Separated Achromatic Doublets

It is also possible to achromatize the focal length of a doublet composed of two widely separated elements of the same glass. Putting it rather succinctly, return to Eq. (6.46) and set $n_{1R} = n_{2R} = n_R$ and $n_{1B} = n_{2B} = n_B$. After a bit of straightforward algebraic manipulation it becomes

$$(n_R - n_B) \left[ (\rho_1 + \rho_2) - \rho_1 \rho_2 d(n_B + n_R - 2) \right] = 0$$

or

$$d = \frac{1}{(n_B + n_R - 2) \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right)}.$$  

Again introducing the yellow reference frequency as we did before, namely $1/f_{1Y} = (n_{1Y} - 1) \rho_1$ and $1/f_{2Y} = (n_{2Y} - 1) \rho_2$, $\rho_1$ and $\rho_2$ can be replaced. Hence

$$d = \frac{1}{n_B + n_R - 2} \frac{f_{1Y} + f_{2Y}}{(n_Y - 1)}$$

where $n_{1Y} = n_{2Y} = n_Y$. Assuming $n_Y = (n_B + n_R)/2$, we have

$$d = \frac{f_{1Y} + f_{2Y}}{2}.$$
or in $d$-light

$$d = \frac{f_{1d} + f_{2d}}{2}.$$  \hfill (6.55)

This is precisely the form taken by the Huygens ocular (Section 5.7.4). Since the red and blue focal lengths are the same, but the corresponding principal planes for the doublet need not be, the two rays will generally not meet at the same focal point. Thus the ocular's lateral chromatic aberration is well corrected but axial chromatic aberration is not.

In order for a system to be free of both chromatic aberrations, the red and blue rays must emerge parallel to each other (no $L \cdot CA$) and must intersect the axis at the same point (no $A \cdot CA$), which means they must overlap. Since this is effectively the case with a thin achromat, it rather implies that multi-element systems, as a rule, should consist of achromatic components in order to keep the red and blue rays from separating (Fig. 6.37). As with all such invocations there are exceptions. The Taylor triplet (Section 5.7.7) is one of these. The two colored rays for which it is achromatized separate within the lens but are recombined and emerge together.

### 6.3.3 Concluding Remarks

For the practical reason of manufacturing ease, the vast majority of optical systems are limited to lenses having spherical surfaces. There are, to be sure, toric and cylindrical lenses as well as many other aspherics. Indeed, very fine, and as a rule very expensive, devices such as high-altitude
reconnaissance cameras and tracking systems may have several aspherical elements. Even so, spherical lenses are here to stay and with them are their inherent aberrations which must satisfactorily be dealt with. As we have seen, the designer (and his faithful electronic companion) must manipulate the system variables (indices, shapes, spacings, stops, etc.) in order to balance out offensive aberrations. This is done to whatever degree and in whatever order is appropriate for the specific optical system. Thus one might tolerate far more distortion and curvature in an ordinary telescope than in a good photographic objective. Likewise, there is little need to worry about chromatic aberration if you want to work exclusively with laser light of almost a single frequency. In any event, this chapter has only touched on the problems (and that more to appreciate than solve them). That they are most certainly amenable to solution is witnessed, e.g. by the remarkable aerial photos of Fig. 6.38 which rather eloquently speak for themselves.

PROBLEMS

6.1* Work out the details leading to Eq. (6.8).

6.2 According to the military handbook MIL-HDBK-141 (23.3.5.3) the Ramsden eyepiece (Fig. 5.81) is made up of two planar-convex lenses of equal focal length $f'$ separated by a distance $2f'/3$. Determine the overall focal length $f$ of the thin-lens combination and locate the principal planes and the position of the field stop.

6.3 Write an expression for the thickness $d$ of a double-convex lens such that its focal length is infinite.

6.4 Suppose we have a positive meniscus lens of radii 6 and 10 and thickness of 3 (any units, as long as you’re consistent) and an index of 1.5. Determine its focal length and the locations of its principal points (compare with Fig. 6.3).

6.5* Show that the determinant of the system matrix in Eq. (6.31) is equal to 1.

6.6 Show that Eqs. (6.36) and (6.37) are equivalent to Eqs. (6.3) and (6.4) respectively.

6.7 Show that the planar surface of a concave-planar or convex-planar lens doesn’t contribute to the system matrix.

6.8 Compute the system matrix for a thick biconvex lens of index 1.5 having radii of 0.5 and 0.25 and a thickness of 0.3 (in any units you like). Check that $|\alpha| = 1$.

6.9* For the lens in the previous problem, determine its focal length and the location of the focal points with respect to its vertices $V_1$ and $V_2$.

6.10 Referring back to Fig. 6.15, show that when $P'P = Rn_2/n_1$ and $P'C = Rn_1/n_2$ all rays originating at $P$ appear to come from $P'$.

6.11 Starting with the exact expression given by Eq. (5.5) show that Eq. (6.40) results, rather than Eq. (5.8), when the approximations for $f_o$ and $f_i$ are improved a bit.

6.12 Supposing that Fig. 6.39 is to be imaged by a lens system suffering spherical aberration only, make a sketch of the image.